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I N S T I T U T E S

O F

NATURAL PHILOSOPHY.



В П Р О С Т О Р

НА П Р О С Т О Р



I N S T I T U T E S

O F

NATURAL PHILOSOPHY,

THEORETICAL AND EXPERIMENTAL.

B Y

W I L L I A M E N F I E L D, LL.D.

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*Omnis Philosophiæ difficultas in eo versari videtur, ut a phænomenis motuum investigemus vires  
Naturæ, deinde ab his viribus demonstremus phænomena reliqua.*

NEWTON.

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1875



T O T H E

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DOCTOR OF LAWS, FELLOW OF THE ROYAL SOCIETY, &c.

I N T E S T I M O N Y O F R E S P E C T F O R A

C H A R A C T E R E M I N E N T L Y D I S T I N G U I S H E D

B Y

C O M P R E H E N S I V E A N D E N L A R G E D V I E W S O F S C I E N C E,

A S S I D U O U S A N D S U C C E S S F U L R E S E A R C H E S I N T O N A T U R E,

A N A R D E N T L O V E O F T R U T H,

I N D E F A T I G A B L E Z E A L I N T H E S E R V I C E O F R E L I G I O N,

S I M P L I C I T Y O F M A N N E R S,

A N D

A N A C T I V E S P I R I T O F P H I L A N T H R O P Y ;

T H I S W O R K

I S I N S C R I B E D,

B Y H I S A F F E C T I O N A T E F R I E N D

A N D O B E D I E N T S E R V A N T,

W I L L I A M E N F I E L D.



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# P R E F A C E.

**N**OTHING can be an adequate apology for obtruding upon the world a new Elementary Work, in a branch of Science already well understood, except the plea of utility. It is wholly upon this ground, that I venture to submit the following Treatise to the public inspection.

The difficulty which I met with, in providing my Classes\* with a Text-book in Natural Philosophy, neither, on the one hand, materially deficient in Mathematical Demonstration, nor, on the other, too copious, or too abstruse, for the purpose of elementary instruction, first suggested the idea of this work. And the apprehension, that others may have met with the same difficulty, induces me to make it public, in hopes that it may be of some use to those who wish to study, or to teach, this Science systematically.

To that class of readers who are satisfied with general views, this work will be of little service. Sketches of philosophy, sufficiently comprehensive to answer their purpose, will easily be found. But the knowledge which is gathered up in this cursory manner, must unavoidably be superficial, and will, in many par-

\* In the Warrington Academy.

ticulars,



ticulars, be confused and inaccurate. What Cicero says of Philosophy in general, is particularly true of Natural Philosophy : *Difficile est enim in philosophia pauca esse ei nota, cui non sint aut pleraque, aut omnia.*\* It may be laid down as an universal maxim, that there is no easy method of attaining excellence. The small portion of learning, or science, which is to be acquired by the help of facilitating expedients, has been justly compared† to a temporary edifice built for a day. It is as unreasonable, to hope to acquire knowledge without undergoing the labour by which it is usually gained, as it would be to expect, that an acorn will become an oak without passing through the ordinary process of vegetation.

All the knowledge of Natural Philosophy which can be acquired by cursory reading, without the assistance of mathematical learning, must consist in an acquaintance with leading facts and general conclusions. To understand the manner in which the laws of nature have been inferred from these facts, and to be able, with certainty and precision, to apply these laws to the explanation of particular phenomena, necessarily requires a previous knowledge of the elements of Geometry, Trigonometry, the Conic Sections, and Algebra. A mechanic who should set about making a machine without the requisite tools, would not act more absurdly, than a student who should attempt to understand the science of Natural Philosophy without these helps. A Preceptor who professes to teach this science in the easy and amusing method of experiment alone, is an architect without his rule, plumb-line, and compasses.

Facts are, it is true, the materials of science; and much praise is unquestionably due to those who have increased the public store, by new experiments accurately made and faithfully related. But it is not in the mere knowledge, nor even in the discovery of facts,

\* Tuscul. Quæst. II. 1.

† Knox on Liberal Education, §. 9.



that philosophy consists. One who proceeds thus far, is an experimentalist; but he alone, who, by examining the nature and observing the relation of facts, arrives at general truths, is a philosopher. A moderate share of industry may suffice for the former: patient attention, deep reflection, and acute penetration, are necessary in the latter. It is therefore no wonder, that amongst many experimentalists there should be few philosophers.

The hardy perseverance, and the vigorous exertions, which are necessary to form this character, are so contrary to that effeminacy and frivolity which distinguish the present age, that, if it were not for the provision which is made in our Universities, and other Seminaries, for the propagation of sound learning of every kind, there would be some reason to apprehend, that all the more abstruse and difficult branches of science would be excluded from the modern system of education, and consequently would fall into disesteem and neglect.

It is by no means the intention of this Treatise to encourage the indolent spirit of the times, by opening a bye-path to the Temple of Philosophy. The known and beaten road is the safest and the best. It has been with the view of assisting the student in his progress, that I have attempted to arrange the leading truths of Natural Philosophy in a perspicuous method, and to demonstrate them with conciseness; adding a brief description of Experiments, adapted to illustrate and confirm the Propositions to which they are respectively subjoined.

Being more desirous to be useful than to appear original, I have freely selected from a variety of Authors such materials as suited my design. Those who are conversant with this class of writers will perceive that, amongst many others, I have made use of the works of NEWTON, Keil, Whiston, Gravesande, Cotes, Smith, Hellsam, Rowning;

*Rowning*, and lastly, *Rutherford*, whose arrangement I have in part adopted.

I have not included in my plan that branch of Philosophy which is properly Chemical, because its object is entirely distinct from that of the Mechanic Philosophy; the province of the latter being to investigate the general laws of nature, and explain their operation in producing particular phenomena; that of the former, to discover the specific differences of bodies: and because the philosophy of chemistry, notwithstanding the great advances which have lately been made in it, is as yet too imperfectly understood, to admit of being digested into a system.

With respect to any inaccuracies or mistakes which may have escaped my attention, I must rely on that candour, which those who are best acquainted with the extent and difficulty of this undertaking will be most inclined to exercise.



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# B O O K I.

## O F M A T T E R.

DEFINITION I. **M**ATTER is an extended and solid substance.

SCHOLIUM. Extension and solidity are discovered to be properties of matter by the senses. Both by the sight and touch we perceive material substances to have length, breadth and thickness, that is, to be extended : and from the resistance which they make to the touch, we acquire the idea, and infer the property, of solidity. It is unnecessary here to inquire whether solidity necessarily supposes impenetrability. Natural Philosophy, being employed in investigating the laws of nature by experiment and observation, and in explaining the phenomena of nature by these laws, has no concern with metaphysical speculations, which are generally little more than unsuccessful efforts, to extend the boundaries of human knowledge beyond the reach of the human faculties.

DEF. II. A body is any portion of matter.

COROLLARY. All bodies have some figure ; for, being portions of matter, they are finite, and therefore bounded by lines either straight or curved.



## P R O P O S I T I O N I.

Matter is infinitely dividible, or is capable of being divided beyond any supposed division.

1. Any particle of matter, placed upon a plane surface, has an upper and a lower side, or a part which touches and another which does not touch the plane, and is therefore dividible.

Plate I.  
Fig. 1.

2. Let CO, MD, be two parallel right lines, to which let AB be drawn perpendicular. In the line MD, on one side of the perpendicular AB, take, at equal distances, the points E, F, G, H. On the other side of AB, in the line CO, take any point C, and join CE, CF, &c. Each of the lines CE, CF, &c. will cut off a portion from AB: but whatever number of lines be drawn in the same manner from C to MD produced, there will still remain a portion of AB not cut off, because no line can be drawn from the point C to the line MD, which shall coincide with CO: the line AB is therefore infinitely dividible.

Plate I.  
Fig. 2.

3. Let the right lines AC and GH be drawn perpendicular to the right line BF. In AC, produced at pleasure, take any points C, C, &c. from which as centers, with the distances CA, CA, &c. describe arcs of circles KAL, NAO, &c. touching BF in the point A, and cutting HG. The farther the central point is taken from A, the greater will be the circles, and the nearer will the arcs approach to the line BF; but the arcs, touching BF in A, cannot touch it in any other point. The line HG is therefore infinitely dividible.

SCHOL. Bodies admit of actual division to a surprising degree of minuteness. A grain of gold may be beaten into a leaf containing 50 square inches, each of which may be divided into 4000 visible parts, that is, the whole grain may be divided into 2,000,000 visible parts. A grain of copper dissolved, is, according to Mr. Boyle, divided into 105,570,000 parts, distinguishable by the sight. Musk, and other odoriferous bodies, will fill a large space with effluvia, without any sensible diminution of their weight. A candle of the smallest size may be seen two miles, that is, disperses light through a sphere whose diameter is four miles, without any sensible diminution.

EXPERIMENT I. A small quantity of solution of copper will tinge a large vessel of water.

2. A small piece of musk will spread its effluvia through a large room.

DEF. III. That force by which the parts of the same body, or of different bodies, on their contact, or near approach, are united to or tend towards each other, is called the *attraction of cohesion*.

P R O P.



P R O P. II.

1. The attraction of cohesion appears in solid bodies.

EXP. I. Observe the different degrees of cohesion in different kinds of wood, suspending weights, from pieces of equal diameter, placed vertically or horizontally, till they break.

2. Measure the different degrees of cohesion in silk, thread, horse hair, &c. by weights suspended from cords of each, placed vertically or horizontally.

The result of sundry experiments to shew the cohesive power of different solids may be seen in the following table, in which each body is taken one tenth of an inch in diameter.

|            |                        |            |                    |
|------------|------------------------|------------|--------------------|
|            | lb                     |            | lb                 |
| Raw Lint   | 37.                    | Ash        | 50, &c.            |
| Horse Hair | 45.                    | Zinc mixed | 18.                |
| Raw Hemp   | 46.                    | Lead       | 29 $\frac{1}{4}$ . |
| Raw Silk   | 53 $\frac{1}{2}$ , &c. | Tin        | 40 $\frac{1}{2}$ . |
| Fir        | 23.                    | Copper     | 299.               |
| Elm        | 35.                    | Brass      | 360.               |
| Alder      | 40.                    | Silver     | 370.               |
| Oak        | 48.                    | Iron       | 450.               |
| Beech      | 50.                    | Gold       | 500.               |

P R O P. III.

2. The attraction of cohesion takes place between particles of the same fluid.

EXP. I. A drop of water, at the end of a small cylinder of wood, will hang in a spherical form. The drop is spherical, because, each particle exerts an equal power in every direction, drawing other particles towards it on every side as far as its power extends.

2. Two globules of mercury, on meeting, unite.

P R O P. IV.

3. The attraction of cohesion takes place between two solid bodies of the same kind; and the more perfect the contact, the greater is the attracting force.

EXP. I. Two plates of glass laid together, though perfectly dry, will cohere.

2. Two lead balls, having each a flat surface scraped smooth, on being forcibly put together, will cohere.

3. Two polished plates of brass, smeared with oil, will cohere strongly.
4. Two plates of lead with equal plane surfaces, heated in boiling water, and immediately put together with tallow upon their surfaces, will cohere so forcibly, as to require a great weight to separate them.

## P R O P. V.

The attraction of cohesion takes place between solids and fluids.

Exp. 1. A plate of glass, or metal, will retain drops of water, or mercury, when inverted.

2. If a plate of glass be in part immersed in a vessel of water, the water which lies contiguous to the glass will rise above the level.

3. Water rises above its level between two parallel plates of glass at a small distance from each other, and in a glass tube having a fine bore, called a capillary tube.

4. The fluid will rise between parallel plates, and in capillary tubes, *in vacuo*. Hence it appears, that the ascent of fluids in capillary tubes is not owing to the pressure of the air.

5. Human blood will rise to a great height in a tube having an exceedingly fine bore.

6. Water will ascend in the cavities of sponge, and other porous bodies.

7. If a drop of oil be poured upon a plate of glass laid horizontally, and another plate of glass be so placed as to meet the first plate at one edge, and be at such a distance from it at the other, as just to touch the drop of oil; this drop, because its touching surface is continually enlarging, will move with increasing velocity, towards that edge. If the planes be lifted up on the side where they meet, the motion will be retarded, stopped, or reversed, according to the degree of elevation.

8. The same phenomenon takes place in a tube of unequal bore.

9. Two glass bubbles, floating near each other, on water, rush together.

10. A glass bubble floating on water in a glass vessel, moves towards the side of the vessel.

SCHOL. These appearances are not owing to any attraction between the bubbles, or between the bubble and the vessel, but to the attraction between water and glass: for no such attraction takes place between the bubble and a glass tube out of the water.

11. Two circular pieces of cork placed upon water, and brought near each other, will be attracted.

12. A piece of wood having a smooth and plane surface, suspended from a beam and balanced, on touching a surface of water, will be attracted; and it will require an additional weight in the opposite scale to separate them.

P R O P.



P R O P. VI.

The heights to which a fluid rises between parallel plates of glass are inversely as the distances of the plates.

The absolute attractive force of the plates, will always remain the same, whatever be the distance of the plates. The same weight of fluid must, therefore, at different distances of the plates, be supported. But the quantity of fluid supported can only continue the same, when the height of the column supported is reciprocally as its base; that is, when as much as the height is increased the base is diminished, and the reverse. Now, the length of the base remaining unvaried, the base can only be made greater or less, by increasing or diminishing the distance between the plates. Therefore, the force, and the quantity of fluid supported, remaining the same, the height will be greater as the distance of the plates is less, and the reverse.

Let  $H, B, D$ , express the height, base and distance, when the plates are at any given distance, and  $h, b, d$ , express the same when they are brought nearer: from what has been shewn,  $H : h :: b : B$ ; but  $b : B :: d : D$ ; therefore  $H : h :: d : D$ .

Exp. Let two parallel plates of glass be immersed, at different distances from each other, in a vessel of coloured water.

P R O P. VII.

The suspension of the fluid, in capillary tubes, is owing to the attraction of the ring of glass contiguous to the upper surface of the fluid.

Every ring of glass below the surface attracts the water above it as much downwards, as it attracts the water below it upwards, and consequently can contribute nothing towards the support of the column: and the action of the lowest ring upon all the fluid in the tube, within its surface of attraction, must either concur with the force of gravity to bring the fluid downwards, or, acting upon it at right angles, can have no effect in suspending it within the tube. The fluid therefore can only be supported by the ring of glass contiguous to its upper surface, which, attracting upwards, opposes the action of gravitation by which the fluid endeavours to descend.—This reasoning may be applied to the fluid raised between parallel plates of glass.

Exp.



EXP. Let a capillary tube be composed of two parts, the bore of one of which is wider than that of the other: immerse its wider orifice in water, till it is filled to any height less than the length of the wider part; the fluid will only rise to the height to which it would rise if the tube were throughout of the same bore with the wider part: but immerse the tube till the fluid enters the smaller part, and the whole column will be suspended, provided its length do not exceed that of the column which a tube of the smaller bore is capable of supporting.

Hence it is manifest, that the water is sustained by the attraction of the narrower part of the tube, for the wider part could not sustain so long a column: it is also manifest, that it is sustained by the ring contiguous to the upper surface; for if it were sustained by the ring at the lower surface, no reason could be assigned why this ring should now support the greater column in both parts of the tube, when it was before only able to sustain a column which filled a part of the wider tube.

Next, let the tube be inverted, and the water be raised into the lower extremity of the wider part; when the suspended column is of greater length than that which a tube of the same bore with the wider part is capable of sustaining, it will immediately sink: whence it is manifest, that the suspension of the column in this case depends upon the attraction of the wider part of the tube; for the narrower part could sustain a larger column: and also, that it is sustained by the ring contiguous to the upper surface; for if it were sustained by the ring at the lower surface, it has been seen that this ring could support a much longer column.

SCHOL. The reason why the narrower or wider ring sustains a column of the same length in the unequal tube above described, as in a tube throughout of the same diameter as the upper ring, is, that the moving forces of the columns are in both cases the same; as will be more fully shewn hereafter.

#### P R O P. VIII.

In capillary tubes, the heights to which the fluid rises are inversely as the diameter of the bores.

The fluid being suspended (Prop. VII.) by the ring of glass contiguous to the upper surface, and the distance to which the attracting force of glass reaches being unvaried; the attracting force which sustains the fluid will be as the number of attracting particles, that



that is, as the circumference, or diameter of the ring, or of the tube. Let  $Q, q$ , then, represent the quantities of fluid to be raised in two tubes of different bores;  $D, d$ , the diameters of their bores; and  $H, h$ , the heights to which fluids rise in the tubes; Because  $Q, q$ , represent two cylinders of the fluid, from the properties of the circle and cylinder (El. XII. 2, 11, and 14.)  $Q : q :: DDH : ddh$ ; and from the nature of this attraction, which is as the diameters of the tubes,  $Q : q :: D : d$ ; therefore  $DDH : ddh :: D : d$ ; and (El. VI. 16.)  $DDHd = ddDh$ , that is,  $DH = dh$ ; consequently  $D : d : h : H$ .

EXP. Let two tubes of different bores be immersed in a vessel of coloured water; it will be found, that the water will rise as much higher in the smaller tube, as the diameter of its bore is less than that of the larger tube.

### P R O P. IX.

Between two glass plates, meeting on one side, and kept open at a small distance on the other, water will rise unequally; and its upper surface will form a curve, in which the heights of the several points above the surface of the fluid, will be to one another reciprocally as their perpendicular distances from the line in which the plates meet.

Let  $AE$  be the surface of the fluid;  $AF$  the line in which the plates meet;  $HL$  the curve formed by the surface of the raised fluid;  $GB, IC, KD, LE$ , perpendiculars to  $AE$ , expressing the heights of the respective points  $G, I, K, L$ , in the curve, above the surface of the fluid, and  $AB, AC, AD, AE$ , perpendiculars to  $AF$ , expressing the distances of the same points from the line in which the plates meet: these heights and distances are reciprocally proportional. For, let the lines  $GB, IC, KD, LE$ , represent pillars of fluid of an equal but exceedingly small breadth: those portions of the glass plates which by their attraction support these pillars, being equal, will sustain equal quantities of fluid; that is, the pillars will be equal. But the pillars may be considered as similar parallelepipeds, which (El. XI. 34.) are equal when their bases and altitudes are reciprocally proportional. And the bases, being equal in breadth, are as their lengths, that is, as the intervals between the plates: and since the intervals continually increase as the distance from the line in which the plates meet increases, these intervals, at the points  $B, C, D, E$ , are as their distances  $AB, AC, AD, AE$ , from the line  $AF$ . Since then the heights of the pillars are reciprocally as the intervals, the heights  $GB, IC, \&c.$  are reciprocally as the distances  $AB, AC, \&c.$

Plate I.  
Fig. 3.

EXP. Let coloured water rise between two glass plates (their inner surfaces being first moistened) meeting on one side according to the proposition.

P R O P.



## P R O P. X.

Some bodies appear to possess a power the reverse of the attraction of cohesion, called repulsion.

EXP. I. If a piece of iron be laid upon mercury, the surface of the mercury near the iron will be depressed.

2. A fine needle laid upon water will swim.

3. Two circular plates of tinfoil being placed upon water, and pressed down by a small additional weight upon their surface, repelling the water, will have a cavity round them: but when they are brought near each other, they will rush together; the re-action of the water on the outer-side of the plates being greater than the <sup>re-</sup>action on the inner-side, where the two cavities produced by repulsion are united.

4. Mercury poured into a recurved glass tube, having the bore on one side exceedingly fine, and on the other large, will not rise so high in the narrow, as in the wide bore: water will rise higher.

5. Melted glass dropped into water forms globules with a stem (called Prince Rupert's drops) which on breaking the stem will burst with great violence and fall into powder.

## P R O P. XI.

All bodies on or near the surface of the earth tend towards its center, by the attraction of gravitation.

A stone, or other heavy body, let fall, will move towards the earth till it meet with some other body to obstruct its course. And bodies move in lines perpendicular to the surface, and consequently in opposite directions on opposite sides of the globe; that is, from the nature of a sphere, in lines which meet at the center. Some bodies ascend, because they meet with resistance, or are acted upon by a force greater than the attraction of gravitation, and in a contrary direction. Vapours, smoke, &c. do not descend, because they are supported by the air.

EXP. I. Smoke or steam will descend in an exhausted receiver.

2. Any boiling fluid being placed in a scale and balanced, the balance will be destroyed by evaporation.

SCHOL.



SCHOL. When we speak of attracting powers, we do not attempt to explain their nature, or assign their causes. Having derived general principles, or laws of nature, from phenomena, we only give a name to these principles, in order to explain other appearances by them.

## P R O P. XII.

Matter is capable of receiving motion from external impulse or force.

This is proved by universal experience.





# B O O K II.

## O F M E C H A N I C S,

O R

### T H E D O C T R I N E O F M O T I O N.

#### C H A P. I.

#### *Of the General Laws of Motion.*

#### PROPOSITION I.

**E**VERY body will persevere in its state of rest, or of motion in a right line, until it is compelled by some force to change its state.

Any body at rest on the surface of the earth will always continue so, if no external force be impressed upon it to give it motion, and if the obstacle which hinders the attraction of gravitation from carrying it towards the center be not removed. A body being put into motion by some external impulse, if all external obstructions were removed, and the attraction of gravitation suspended, would move on for ever in a right line; for there would be no cause to diminish the motion, or to alter its direction. This cannot be fully established by experiment, because it is impossible intirely to remove all obstructions: but, since the less obstruction remains the longer motion continues, it may be reasonably inferred, that if all obstacles could be removed, motion once communicated to any body would never cease.

Exp. 1. A ball at rest requires some degree of force to put it into motion; and when in motion, it will continue to move longer on a smooth surface than on a rough one.

2. If a stone be whirled round in a string, on being set at liberty it will continue to move with the force which it has acquired.

3. If a vessel containing a quantity of water be moved along upon an horizontal plane, the water resisting the motion of the vessel, will at first rise up in the direction contrary to that in which the moving force acts: when the motion of the vessel is communicated to the water it will persevere in this state; and if the vessel be suddenly stopped, resisting the change from motion to rest, it will rise up on the opposite side. In like manner, if a horse which was standing still, suddenly starts forwards, the rider will be in danger of being thrown backwards, if the horse stops suddenly, the rider will be thrown forwards.

### P R O P. II.

The change of motion produced in any body is proportional to the force impressed, and in the direction of that force.

If a given force will produce a given motion, a double force will produce the double of that motion. If a new force be impressed upon a body in motion, in the direction in which it moves, its motion will be increased proportionally to the new force impressed: if this force acts in a direction contrary to that in which the body moves, it will lose a proportional part of its motion: if the direction of this force be oblique to the direction of the moving body, it will give it a new direction.

Exp. Let one clay ball, suspended by strings, strike another clay ball suspended in the same manner, at rest or in motion, it will communicate a degree of motion greater or less in proportion to the force of the striking body: in the opposite direction motion will be destroyed in the same proportion.

### P R O P. III.

To every action of one body upon another, there is an equal and contrary re-action: or, the mutual actions of bodies on each other are equal and in contrary directions.

Whatever quantity of motion any body communicates to another, or whatever degree of resistance it takes away from it, the acting body receives the same quantity of motion, or loses the same degree of resistance, in the contrary direction: the resistance of the body acted upon producing the same effect upon the acting body, as would have been produced by an active force equal to, and in the direction of, that resistance.

COR.



COR. Hence it appears, that one body acting upon another loses as much motion as it communicates; and that the sum of the motions of any two bodies in the same line of direction cannot be changed by their mutual action.

EXP. Let a clay ball in motion, strike another equal to it at rest: the striking body will lose half its quantity of motion, which will be communicated to the other body.

SCHOL. Although these laws of nature may be illustrated by experiments, their best confirmation arises from hence, that all the particular conclusions drawn from them agree with universal experience.

## C H A P. II.

*Of the Comparison of uniform Motions.*

## P R O P. IV.

The quantities of matter in bodies are in the compound ratio of their magnitudes and densities.

If the magnitudes of two bodies be given, the quantities of matter will be as the densities: if their densities be given, the matter will be as the magnitudes: therefore the matter is universally in the compound ratio of the magnitudes and densities. For example, If A and B be two balls equal in magnitude, the quantity of matter in A will be to that in B, as the density of A is to that of B: if both be of the same density, their quantities of matter will be as their magnitudes.

## P R O P. V.

The velocities with which bodies move, are directly as the spaces they describe, and inversely as the times in which they describe these spaces.

It is manifest, that the degree of velocity increases as the space a body passes over in a given time increases, and as the time in which it passes over a given space decreases; and the reverse. For example, If one ball A move over six feet, and another ball B over three feet in the same time, A has double the velocity of B: but if the ball A passes over six feet in  
two

two seconds of time, and the ball B passes over six feet in one second, the velocity of B is double of that of A.

P R O P. VI.

The spaces which bodies describe are in the compound ratio of their times and velocities.

It is evident, that the longer time any body continues to move, and the greater velocity it moves with, the greater space it will pass through; and the reverse. If, for example, the body A moves for one second, and the body B moves for two seconds, both moving with the same velocity, A will move through ~~half~~ <sup>half</sup> as much space as B: if A moves with two degrees of velocity, and B with one degree of velocity, A will in the same time pass over twice as much space as B.

P R O P. VII.

The times in which bodies move are directly as the spaces, and inversely as the velocities.

The greater space any body passes through, and the less degree of velocity it moves with, the greater must be the portion of time taken up in the motion; and the reverse. For example, If the ball A moves with the same velocity with the ball B, but passes over double the space, A will move twice as long as B: if A moves over the same space with B, and with half the velocity, it must, in this case also, move twice as long as B.

P R O P. VIII.

The force required to move a body at rest is as the quantity of matter to be moved.

Each particle of matter in any body resisting motion, a force must be exerted upon each particle to overcome this resistance; if therefore two bodies containing different quantities of matter are to be moved, the greater body will require the greater force.

DEF. I. The *momentum* of any body is its quantity of motion.

P R O P.



## P R O P. IX.

In moving bodies, if the quantities of matter be equal, the momenta will be as the velocities.

It is manifest, that if the body A be equal to the body B, but A has twice the velocity of B, A has twice as much motion as B.

## P R O P. X.

The velocities of two bodies being equal, their momenta will be as their quantities of matter.

The bodies A and B moving with equal velocities, since every portion of matter in A has as much motion as an equal portion of B, it is evident, that if A has twice the quantity of matter in B, it must have twice as much motion.

## P R O P. XI.

The momenta of moving bodies are in the compound ratio of their quantities of matter and velocities.

The greater quantity of matter there is in any body, and the greater velocity it moves with, the greater will evidently be its quantity of motion; and the reverse. If, for example, the body A be double of the body B, and moves with twice its velocity, the momentum of A will be quadruple of that of B: for it will have twice the momentum of B from its double velocity, and also twice the momentum of B from its double quantity of matter.

COR. Hence, if in two bodies the product of the quantities of matter and velocities are equal, their momenta are equal.

## P R O P. XII.

The velocities of moving bodies are as their momenta directly, and their quantities of matter inversely.

The greater momentum any body has, and the less quantity of matter it contains, the greater must be its velocity. For example, If the body A is half of B, and their momenta are equal, A will move with twice the velocity of B; and if A and B are equal, and the momentum of A is double of that of B, its velocity will also be double.

PROP.

## P R O P. XIII.

The force, or power of overcoming resistance, in any moving body, is as its momentum.

Since a body having a certain degree of motion is able to overcome a certain degree of resistance, it is manifest, that with an increased momentum, it will be able to overcome a greater resistance.

COR. Hence the momentum of any body is measured by its power of overcoming resistance.

SCHOL. I. Let  $\mathcal{Q}, q$ , denote the quantities of matter in any two bodies,  $D, d$ , their densities, and  $B, b$ , their bulk or magnitude,  $V, v$ , their velocities,  $T, t$ , the times of their motion,  $S, s$ , the spaces over which they pass,  $P, p$ , the moving powers,  $M, m$ , their momenta, and  $F, f$ , their force: the preceding propositions may be thus expressed:

$$\text{PROP. IV. } \mathcal{Q} : q :: BD : bd$$

$$\text{V. } V : v :: \frac{S}{T} : \frac{s}{t}$$

$$\text{VI. } S : s :: TV : tv$$

$$\text{VII. } T : t :: \frac{S}{V} : \frac{s}{v}$$

$$\text{VIII. } P : p :: \mathcal{Q} : q$$

$$\text{IX. } M : m :: V : v : \text{ if } \mathcal{Q} = q.$$

$$\text{X. } M : m :: \mathcal{Q} : q : \text{ if } V = v.$$

$$\text{XI. } M : m :: \mathcal{Q}V : qv$$

$$\text{XII. } V : v :: \frac{M}{\mathcal{Q}} : \frac{m}{q}$$

$$\text{XIII. } F : f :: M : m$$

SCHOL. 2. The principal analogies of velocities, spaces, times, momenta, quantities of matter, and forces are expressed in the following table:

$$f : m : qv : \frac{qs}{t}$$

$$m : f : qv : \frac{qs}{t}$$

$$s : tv : \frac{tm}{q} : \frac{tf}{q}$$

$$v : \frac{m}{q} : \frac{s}{t} : \frac{f}{q}$$

$$t : \frac{s}{v} : \frac{sq}{m} : \frac{sq}{f}$$

$$q : \frac{m}{v} : \frac{f}{v} : \frac{mt}{s} : \frac{ft}{s}$$

SCHOL.



SCHOL. 3. It has been maintained by several philosophers, that the momentum of any moving body, is not simply as its quantity of matter multiplied into its velocity, but as its quantity of matter multiplied into the square of its velocity. But it is a sufficient refutation of this doctrine, to observe, that since the force required to destroy any motion must be the proper measure of the quantity of that motion; and since it is found from experience, that the motion of any body will be destroyed, if a body having the same quantity of matter and the same velocity act upon it in the opposite direction; the force destroyed, that is the momentum, must have been as the quantity of matter multiplied into the velocity of the body.

## C H A P. III.

*Of the Composition and Resolution of Forces.*

## P R O P. XIV.

**A** BODY acted upon by two forces united, will describe the diagonal of a parallelogram, in the same time in which it would have described its sides by the separate action of these forces.

If, in a given time, a body, by the single force M impressed upon it at the point A, would be carried from A to B; and by another single force N impressed upon it at the same point, would be carried from A to C; complete the parallelogram ABDC; and with both forces united, the body will be carried in the same time through the diagonal of the parallelogram from A to D. For since the force N acts in the direction of the right line AC parallel to BD, this force (by Prop. II.) has no effect upon the velocity with which the body approaches towards the line BD by the action of the force M. The body will therefore arrive at the line BD in the same time, whether the force N is impressed upon it or not; and at the end of that time will be found somewhere in the line BD. For the same reason, at the end of the same time it will be found somewhere in the line CD; therefore it must be found at the point D, the intersection of these two lines. And (by Prop. I.) it will move in a right line from A to D.

Plate 1.  
Fig. 4.

EXP. Two equal lead weights suspended at the end of a triangular frame of wood to give them a steady motion, and let fall at the same instant from equal heights, striking a ball suspended by a cord at the point in which their lines of direction meet, will carry it forwards in the diagonal of the parallelogram of those lines produced.

COR. I. Hence, the velocity produced by the joint action of two forces is to that with which the body moves by the action of each force singly, as the diagonal of  
D the



the parallelogram to either side; for the diagonal is described in the same time with either side.

COR. 2. If two sides of a triangle represent the directions and quantities of two forces, the third side will represent the direction and quantity of a force equivalent to both acting jointly: for the third side may be considered as the diagonal of a parallelogram.

COR. 3. A body may be moved through the same line by different pairs of forces. In plate 1. fig. 4. AD is the diagonal both to the parallelogram ABCD, and to the parallelogram AEDF; and consequently expresses a force equal to AB, AC, and to AF, AE.

### P R O P. XV.

The velocity produced by two joint forces, when they act in the same direction, will be as the sum of the forces, and when they act in opposite directions, will be as their difference; and the velocity will be the greater the nearer they approach to the same direction, and the reverse.

Plate 1.  
Fig. 5.

In the parallelograms ABCD, in which AB, AC, express the direction and quantity of two joint forces, the side AB being placed at different angles with AC, it is manifest that as AB approaches towards AC, the diagonal increases, till at length it becomes equal to AC, CD, that is, to AC, AB, and the velocity is as the sum of the forces, since they act in the same direction.

Plate 1.  
Fig. 6.

In the parallelograms ABCD, as AB recedes from CD, the diagonal decreases, till at length it vanishes with the angle, and the two sides AB, AC, constitute one right line, the parts of which, AB, AC, representing forces acting in opposite directions, if the forces be equal, they will destroy each other; if unequal, the velocity will be as their difference.

### P R O P. XVI.

Any single force or motion may be resolved into two forces or motions; and the directions of these may be infinitely varied: also any two forces may be compounded into single forces.

Plate 1.  
Fig. 4.

A body moving in the line AD, may be considered as receiving its direction and velocity from two forces acting jointly in the directions AB, AC, or from two other forces expressed by AF, AE: for (Prop. XIV. Cor. 3.) each pair would produce the same effect.

In



In like manner the direction and quantities of the forces will be diversified with every change of the sides of the parallelogram, the diagonal remaining the same.

It is also manifest, that any two joint forces may be compounded into one, being expressed by the sides of a parallelogram, or its diagonal.

## P R O P. XVII.

If a body is acted upon by three forces, which are proportional to, and in the direction of, the three sides of a triangle, the body will be kept at rest.

Let a body placed at D be acted upon by three forces AD, GD, FD, proportional to, and in the direction of the three sides of the triangle GED: complete the parallelogram GEFD; and make AD equal to, and in the direction of the diagonal ED.

Plate 1.  
Fig. 7.

If the body at D be acted upon by the forces AD, ED, equal and in opposite directions, it will be kept at rest. But the force ED (Prop. XVI.) is equivalent to the two forces DG, DF, that is, DG, GE; therefore the body acted upon by the three forces AD, DG, DF, that is, by three forces proportional to, and in the direction of the sides of the triangle GED, will be at rest.

EXP. Let three weights in the proportions of 3, 4, 5, be suspended from cords which pass over pulleys and meet in a point; if the directions of the cords be parallel to the sides of a triangle, (drawn in a plane parallel to the plane of the cords) whose sides are to each other as the weights, a ball at the point in which the cords meet will be kept at rest.

COR. The body will be at rest if the three forces are proportional to the three sides of a triangle drawn perpendicular to the direction of the forces; for such a triangle is similar to the former.

## P R O P. XVIII.

The force of oblique percussion is to that of direct or perpendicular action, as the sine of the angle of incidence to radius.

Let a body strike upon the plane AD, at the point D, in the direction BD: the line BD expressing the force of direct impulse may be resolved into two others, BC, BA, the one parallel, the other perpendicular to the plane: of these, the force BC, parallel to the plane, cannot affect it: the whole force upon the plane may therefore be expressed by BA. But BA is to BD as the sine of the angle of incidence BDA is to radius.

Plate 1.  
Fig. 8.

SCHOL. If the surface to be struck be a curve, let AD be made tangent to the curve at D, and the proof will be the same.



## P R O P. XIX.

The force of oblique action produced by percussio is to that of direct action, as the cosine of the angle comprehended between the direction of the force and that in which the body is to be moved, to radius.

Plate I.  
Fig. 9.

Let FD represent a force acting upon a body at D, and impelling it towards E; but let DM be the only way in which it is possible for the body to move. The force FD may be resolved into two forces FG, FH or GD; of which only the force GD impels it towards M. And, FD being radius, GD is the cosine of the angle FDG, or MDE, comprehended between the direction of the force, and that in which the body is to be moved.

## C H A P. IV.

*Of Motion as communicated by Percussion in Non-Elastic, and Elastic Bodies.*

DEF. II. Bodies are non-elastic, which, when one strikes another, do not rebound, but move together after the stroke.

COR. Hence their velocities after the stroke are equal.

DEF. III. Bodies are elastic, which have a certain spring, by which their parts, upon being pressed inwards by percussio, return to their former state, throwing off the striking body with some degree of force: when the elasticity is perfect, the body restores itself with a force equal to that with which it is compressed.

EXP. The existence of this property is visible in a ball of wool, cotton, or sponge compressed.

## P R O P. XX.

When one non-elastic body in motion, strikes upon another at rest, or moving with less velocity in the same direction, the sum of their momenta remains the same after the stroke as before.

For



For (Prop. III. Cor.) as much motion as the striking body communicates, so much it loses; whence, if the motions of the bodies are in the same direction, whatever is added to the motion of the preceding body will be subducted from that which follows, and the sum will remain the same.

## P R O P. XXI.

When two non-elastic bodies, moving in an opposite direction strike upon each other, the sum of their momenta after the stroke, will be equal to the difference of their momenta before the stroke.

For (from Prop. III. Cor.) that body which had the least motion will destroy a quantity equal to its own in the other; after which, they will move together with the remainder, that is, the difference.

Exp. Let two cylinders filled with clay, A, B, be of equal weight, and suspended by cords from equal heights; let two other cylinders of the same kind, C, D, but in weight as 2 to 1, be suspended from the same height. The heights from which they are let fall, in the arc formed by the motion of the cylinder (from the nature of the pendulum, afterwards to be explained) will be the measure of their velocity; and (by Prop. XI.) their momenta will be as their velocities multiplied into their quantities of matter; whence the cases of the two preceding propositions may be established by the following experiments. N. B. Quantity of matter is expressed by  $q$ , velocity by  $v$ , and momentum by  $m$ .

No. 1. Prop. XX. Case 1. Let the cylinder A fall from the height of 18 inches, upon the cylinder B at rest. The momentum of A before the stroke (by Prop. XI.) is 18; for the quantity of matter is 1, and the velocity 18; whence  $q\ 1 \times v\ 18 = m\ 18$ . After the stroke, the quantity of matter being (Def. II.) 2, and the velocity of each cylinder 9, the momentum will be 18:  $q\ 2 \times v\ 9 = m\ 18$ .

No. 2. Case 2. Let A fall from 18 inches, and B from 9, in the same direction; their momenta before the stroke are  $18 + 9 = 27$ ; after the stroke, the quantity of matter will be 2, and the velocity  $13\frac{1}{2}$ ; whence  $v\ 13\frac{1}{2} \times q\ 2 = m\ 27$ .

No. 3. Prop. XXI. Case 1. Let the equal cylinders A and B fall in opposite directions, from the height of 12 inches; the momenta being equal and opposite, the motion of both will be destroyed.

No. 4. Case 2. Let A fall from the height of 12 inches, and meet B falling in the opposite direction from 6 inches; their velocity after the stroke being 3, and quantity of matter 2, the momentum will be 6;  $q\ 2 \times v\ 3 = m\ 6$ .

Prop.



No. 5. Prop. XX. Case 3. Let the cylinder C, double of the cylinder D, fall from 12 inches on D at rest. Before the stroke, the quantity of matter in C is 2, and its velocity is 12; whence its momentum is 24;  $q\ 2 \times v\ 12 = m\ 24$ . After the stroke, the velocity will be 8, and quantity of matter 3; whence  $q\ 3 \times v\ 8 = m\ 24$ .

No. 6. Case 4. Let C fall from 12 inches, and D from 6 inches in the same direction. Before the stroke, the velocity of C is 12, and quantity of matter 2; whence its momentum is 24;  $q\ 2 \times v\ 12 = m\ 24$ ; and the velocity of D is 6, and its quantity of matter 1; whence  $q\ 1 \times v\ 6 = m\ 6$ ; therefore the whole momentum is 30. After the stroke, the velocity of the whole is 10, and the quantity of matter 3; whence  $q\ 3 \times v\ 10 = m\ 30$ .

No. 7. Prop. XXI. Case 3. Let C fall from 6 inches, and D from 12, in opposite directions, the quantity of matter in C being 2, and its velocity 6; and the quantity of matter in D being 1, and its velocity 12, their momenta will be equal, and being opposite, will destroy each other.  $C\ q\ 2 \times v\ 6 = m\ 12$ ;  $D\ q\ 1 \times v\ 12 = m\ 12$ .

No. 8. Case 4. Let C fall from 3 inches, and D from 12, in opposite directions: before the stroke the momentum of C is 6;  $q\ 2 \times v\ 3 = m\ 6$ , and the momentum of D is 12;  $q\ 1 \times v\ 12 = m\ 12$ ; whence the difference of their momenta is 6. After the stroke, the velocity is 2, and quantity of matter 3; whence the momentum is 6;  $q\ 3 \times v\ 2 = m\ 6$ .

### P R O P. XXII.

When one elastic body strikes upon another of the same kind, the one loses, and the other gains, twice as much momentum, as if the bodies had been void of elasticity.

For, since (by Def. III.) perfectly elastic bodies, on percussio, restore themselves with a force equal to that with which they are compressed, whatever momentum is gained by one body, or lost by the other, on percussio, from the law of re-action, the same must be gained, or lost, from the power of elasticity.

COR. I. Hence the momentum of elastic bodies after percussio may be found, by doubling the momentum which would have remained, had the bodies been non-elastic.

2. If one of the bodies, considered as non-elastic would lose more than half its momentum, as elastic it loses more than all, that is, acquires a negative momentum in a contrary direction.

EXP. The following experiments may be made with ivory balls suspended from strings: they correspond with the preceding experiments on non-elastic bodies.

Let



Let A and B be equal balls; and let C be a ball double of the ball D.

No. 1. A, falling from 18 inches on B at rest, has 18 degrees of momentum before the stroke: therefore, after the stroke, supposing the balls non-elastic, the same momentum belonging to the two equal balls together, each has 9 degrees of momentum, and A has lost and B gained 9. This being doubled, A, as elastic, will lose 18, and B will gain 18 degrees of momentum: whence A will be at rest, and B will move with 18 degrees of momentum.

No. 2. A, falling from 18 inches, and B from 9 in the same direction; as non-elastic, after the stroke, each has  $13\frac{1}{2}$  momentum, or A has lost  $4\frac{1}{2}$ , and B gained  $4\frac{1}{2}$ . As elastic, after the stroke, A loses 9, B gains 9; therefore A rises to 9 inches, B to 18.

No. 3. A and B, falling in opposite directions from 12 inches, as non-elastic, would lose all their momentum: as elastic, each loses 24 degrees of momentum; that is, gains 12 in the contrary direction.

No. 4. A, falling from 12 inches, and B in the opposite direction from 6, as non-elastic, the momentum of each, after the stroke, will be in the direction of A; whence A loses 9, and B loses 9, moving 3 degrees in the contrary direction. As elastic, A loses 18, or has 6 in the contrary direction, and B loses 18, or gains 12 in the contrary direction.

No. 5. C, double of D, falling from 12 inches on D at rest, the momentum of C before the stroke being 24, and of D nothing; as non-elastic, C after the stroke, having its momentum 16, and moving with the velocity 8, will have lost 4 degrees of velocity, and 8 of momentum: and D will have gained 8 of each. As elastic, therefore, C will lose 8 degrees of velocity, or (Prop. XI.) 16 of momentum, and D will gain 16 of each; that is, C will move with 4 degrees of velocity, and D with 16.

No. 6. C falling from 12 inches, and D from 6 in the same direction, before the stroke, the velocity of C is 12, and its momentum 24; and the momentum of D 6. After the stroke, as non-elastic, the momentum of C is 20, because  $q 2 \times v 10 = m 20$ ; and the momentum of D is 10, because  $q 1 \times v 10 = m 10$ ; therefore C has lost 4 degrees of momentum, or 2 degrees of velocity, and D gained 4 of each. If, therefore, the gain or loss be doubled on account of the elasticity of C and D, C will lose 8 degrees of momentum, or 4 of velocity, and D will gain 8 of each; that is, C will move with 8 degrees of velocity, and D with 14.

No. 7. C falling from 6 inches, and D from 12 in opposite directions, their momenta being equal, would destroy each other without elasticity: therefore, being elastic, each will acquire the momentum of 12 in opposite directions; that is, D will return to 12, and C to 6.

No. 8. C falling from 3 inches, and D from 12 in opposite directions; since the momentum of C before the stroke is 6, and of D 12, as non-elastic bodies they would,  
after



after the stroke, move in the direction of D, with the velocity 2; whence C would move in the direction contrary to its first motion with 4 degrees of momentum, and lose 10; and D would lose 10: therefore, being elastic, C will lose 20 degrees of momentum, and also D 20; whence C will move in the contrary direction with 14 degrees of momentum; that is, will return to 7; and D will return to 8.

COR. 1. If the sum of two conspiring momenta, or the difference of two contrary momenta, be divided by the sum of the quantities of matter in both the moving bodies, the quotient will give the common velocity after the stroke.

SCHOL. Let  $A$  and  $B$  be two spherical bodies, moving with their centers in the same line; and let their velocities be  $a$  and  $b$ . The momentum of  $A$  before the stroke is  $Aa$ , and that of  $B$  is  $Bb$ ; the sum, or their difference is  $Aa + Bb$ , or  $Aa - Bb$ . Therefore (by Prop. XX. and XXI.) the momentum after the stroke is expressed by  $Aa + Bb$ , and, their common velocity by  $\frac{Aa + Bb}{A + B}$ . Hence the momentum of  $A$  after the stroke is  $\frac{AAa + ABb}{A + B}$ ; and that of  $B$  is  $\frac{ABa + BBa}{A + B}$ .

Next, suppose the bodies perfectly elastic. Subtract the momentum of  $A$  considered as non-elastic, after the stroke,  $\frac{AAa + ABb}{A + B}$ , from its momentum before the stroke,  $Aa$ ; and the remainder,  $\frac{ABa + ABb}{A + B}$ , will express the momentum in that case lost by  $A$ , and gained by  $B$ . Subtract this remainder,  $\frac{ABa + ABb}{A + B}$ , from the momentum of  $A$ , as non-elastic, after the stroke,  $\frac{AAa + ABb}{A + B}$ ; and add the same remainder to the momentum of  $B$  after the stroke,  $\frac{ABa + BBb}{A + B}$ : the difference,  $\frac{AAa + 2ABb - ABa}{A + B}$ , will express the momentum of  $A$  after the stroke, and the sum  $\frac{2ABa + BBb + ABb}{A + B}$ , will express the momentum of  $B$  after the stroke, supposing them perfectly elastic. And  $\frac{Aa + 2Bb - Ba}{A + B}$ , and  $\frac{2Aa + Bb + Ab}{A + B}$ , will express their respective velocities.

COR. 2. If there be any number of elastic, equal, and spherical bodies whose centers are placed in the same line, and the first body strikes upon the second in the direction of that line, all the bodies will be at rest except the last, which will move off with the velocity of the first.

EXP. Several equal ivory balls being so suspended as to have their centers in a right line, if the first be let fall upon the second, the last will fly off, to the height from which the first fell.

COR. 3.



COR. 3. When the striking ball is less than the quiescent, there will be an increase of momentum.

EXP. Let the ball D fall from 12 inches upon C, double of D, at rest. If they were non-elastic, they would proceed together, and, their velocity being the same, C, after the stroke, would have double the momentum of D; that is, C would have 8 degrees and D 4; whence D would have communicated more than half its momentum to C. The effect being doubled by the elasticity of the bodies, D communicates to C 16 degrees of momentum, and loses as much itself, or returns with 4 degrees of momentum in the contrary direction; while C moves forwards with 4 degrees more momentum than D had at the first. Thus the whole sum of momentum is increased from 12 to 20 degrees: but as much as the momentum is increased in the direction in which D first moved, so much is given to D in the contrary direction. In this manner may momentum be continually increased by a series of bodies.

COR. 4. If a non-elastic body strikes upon an immoveable obstacle, it will lose all its motion; an elastic body will return with a force equal to the stroke.

EXP. Let a lead ball, and an ivory ball, strike upon any fixed plane.

## C H A P. V.

### *Of Motion as produced by the Attraction of Gravitation.*

## S E C T. I.

### *Of the Laws of Gravitation in Bodies falling without Obstruction.*

## P R O P. XXIII.

The motion of a body falling freely by the attraction of gravitation is uniformly accelerated, or its velocity increases equally in equal times.

A new impression being made upon the falling body, at every instant, by the continued action of the attraction of gravity, and the effect of the former (by Prop. I.) still remaining,  
E
the

the velocity must continually increase. Suppose a single impulse of gravitation, in one instant, to give it one degree of velocity; if after this the force of gravitation were entirely suspended, the body would continue to move with that degree of velocity, without being accelerated or retarded. But, because the attraction of gravitation continues, it produces as great a velocity in the second instant as in the first; which being added to the first, makes the velocity in the second instant double of what it was in the first. In like manner, in the third instant, it will be tripled; quadrupled in the fourth; and in every instant, one degree of velocity will be added to that which the body had before; that is, the motion will be uniformly accelerated.

COR. The velocities of falling bodies, are as the times in which they are acquired.

EXP. Observe the continual acceleration of a falling ball.

#### P R O P. XXIV.

The force of the attraction of gravitation acting upon any body is as its quantity of matter.

For, each particle of matter in any body being acted upon by gravitation, the greater number of particles are contained in any body, the greater force must be exerted upon it; that is, the force increases as the quantity of matter increases.

EXP. Let two unequal balls, suspended by threads of the same length, be let fall at the same time from points equally distant from the lowest points of the arcs in which they move: the vibrations of each will be performed in equal times, and consequently their velocities will be equal; whence the momenta (Prop. XI.) will be as the quantities of matter: but (by Prop. XIII.) the force producing motion, is as the quantity of motion, or momentum, produced: therefore the force of gravitation is as the quantity of matter; that is, as much greater force is exerted upon the larger body than upon the less, as its quantity of matter is greater than that of the less.

COR. The weight of any body is as its quantity of matter; for weight is the degree of force with which any body is acted upon by gravitation.

#### P R O P. XXV.

The velocities of bodies falling from the same height without resistance, are equal.

If two bodies of different quantities of matter fall from the same height, the attracting force which acts upon the greater body, will (Prop. XXIV.) exceed that which acts upon



upon the lefs, as much as the greater body exceeds the lefs in quantity of matter; whence they muft move with equal velocities.

Exp. A guinea, and a feather, or other light body, in the exhausted receiver of an air pump, will fall through the fame space in the fame time.

P R O P. XXVI.

The fpaces described by falling bodies are as the fquares of the times from the beginning of the fall, and alfo as the fquares of the laft acquired velocities.

In the triangle ABC, let AB exprefs the time in which a body is falling, and BC the velocity which it has acquired at the end of the fall; let AF, AD, be parts of the time AB; and through F, D, draw FG, DE, parallel to BC.

Plate 1.  
Fig. 10.

Because the triangles ABC, ADE, are fimilar, AB is to AD as BC to DE; but AB and AD exprefs times of defcent, and BC expreffes the velocity acquired in the time AB; therefore, fince (Prop. XXIII. Cor.) the velocities are as the times, DE expreffes the velocity acquired in the time AD. In like manner, GF, any other right line parallel to BC, expreffes the velocity acquired in the time AF. Therefore the fum of the lines which may be fupposed drawn parallel to CB in the triangle ADE; that is, the whole triangle ADE, will represent the fum of the feveral velocities with which the falling body moves in the time AD. For the fame reafon, the triangle ABC will represent the fum of the velocities with which the falling body moves in the time AB. Since therefore it is manifef, that the fpace which a body paffes through in any moment of time is as the velocity with which it moves at that moment; and confequently, that the fpaces through which it paffes in any times whatfoever, are as the fums of the velocities with which it moves in the feveral moments of thofe times; the fpaces paffed through in the times AD, AB, are to each other as the triangles ADE, ABC. But the triangle ADE (El. VI. 19.) is to the triangle ABC in the duplicate ratio of the homologous fides AD, AB, and alfo of DE, BC; that is, the fpaces are as the fquares of the times, and alfo as the fquares of the laft acquired velocities.

Exp. Let there be two pendulums, one of which vibrates twice as faft as the other, a ball let fall from fuch a height above the ball of the fhorter pendulum as to reach it in one vibration, muft fall from four times this height, to reach the longer pendulum in one of its vibrations.

Cor. I. The times in which bodies fall from unequal heights, and their laft acquired velocities, are as the fquare roots, or in the fubduplicate ratio of their heights. If the height or fpace be called S, the velocity V, and the time T; fince TT is as S, T will be as  $\sqrt{S}$ ; and fince VV is as S, V will be as  $\sqrt{S}$ .



COR. 2. If the time of the fall of a body be divided into equal parts, the spaces through which it falls in each of these parts, taken separately, will be as the odd numbers 1, 3, 5, &c. The spaces being as the squares of the times or velocities, if the times be as the numbers 1, 2, 3, 4, the spaces will be as 1, 4, 9, 16; whence, in the first time the space will be as 1, in the second time, the space passed over will be as 3, in the third, as 5, &c.

## P R O P. XXVII.

The space which a body passes over in any given time from the beginning of the fall, is half that which it would pass over in the same time, moving with the last acquired velocity.

Plate 1.  
Fig. 10.

For the triangle ABC (by Prop. XXVI.) expresses the space passed over in the time AB, when the motion is uniformly accelerated; the last acquired velocity is expressed by BC; and the rectangle of AB, BC, rightly expresses the space passed through in the time AB with the equable velocity BC: since therefore the triangle ABC is half of the rectangle AB, BC, the proposition is manifest.

## P R O P. XXVIII.

The motion of a body thrown upwards is uniformly retarded by gravitation: the time of its rise will be equal to that in which a body falling freely acquires the same degree of velocity with which it is thrown up; and the height to which it rises will be as the square of the time, or first velocity.

The same force which accelerates a falling body, acting in an opposite direction upon one thrown upwards, must retard it: and, since the action of gravitation is uniform, in whatever time it generates any velocity in a falling body, it must in the same time destroy the same velocity in a rising body: through whatever space the falling body must pass to acquire any velocity, the rising body must pass through the same, to lose it: and whatever ratio the spaces bear to the velocities and times in one case, must take place in the other: the effect of gravitation in rising bodies being in all respects the reverse of its effect upon falling bodies.

S E C T.



## S E C T. II.

*Of the Laws of Gravitation in Bodies falling down inclined Planes.*

DEF. IV. An inclined plane, is a plane which makes an acute or obtuse angle with the plane of the horizon.

## P R O P. XXIX.

The motion of a body descending down an inclined plane, is uniformly accelerated.

In every part of the same plane, the accelerating force has the same ratio to the force of gravitation acting freely in a perpendicular direction, and is therefore (El. V. 9.) equally exerted in every instant of the descent; whence, (as was shewn concerning bodies falling freely, Prop. XXIII.) the motion must be uniformly accelerated.

COR. Hence, whatever has been demonstrated concerning the perpendicular descent of bodies, is equally applicable to their descent down inclined planes, the motion in both cases being uniformly accelerated by the same power of gravitation.

EXP. 1. Observe the acceleration of motion down an inclined plane.

EXP. 2. Measure the spaces passed over by a body down an inclined plane by the vibration of pendulums, as in Prop. XXVI. Exp.

## P R O P. XXX.

The force with which a body descends by the attraction of gravitation down an inclined plane, is to that with which it would descend freely, as the elevation of the plane to its length.

Let AB be the length of an inclined plane, and AC its elevation, or perpendicular height. If the force of gravitation with which any body descends perpendicularly be expressed by AC, and this force be resolved into two forces, AD, DC, by drawing CD perpendicular to AB; because the force CD is destroyed by the re-action of the plane, the body descends down the inclined plane only with the force AD. And (El. VI. 8. Cor.) AD is to AC, as AC is to AB; that is, the force of gravitation down the inclined plane is to the same force acting freely, as the elevation of the plane to its length.

COR.

Plate I.  
Fig. II.



COR. Hence, the force necessary to sustain a body on an inclined plane, is to the absolute weight of a body, or its unobstructed force, as the elevation of the plane to its length: for the force requisite to sustain a body, must be equal to that with which it endeavours to descend; which has been shewn to be to that with which it would descend freely, as the elevation of the plane to its length.

EXP. A weight of 4 ounces laid upon a wheel carriage, so balanced as just to move freely on an inclined plane, the length of which is to its elevation as 2 to 1, will be balanced by a weight of two ounces, suspended freely from a cord, passing over a pulley parallel to the plane.

### P R O P. XXXI.

The space described in any given time by a body descending down an inclined plane, is to the space through which it would fall perpendicularly in the same time, as the elevation of the plane to its length.

Plate I.  
Fig. II.

Let AC represent the force with which a body would fall perpendicularly: CD being drawn from C perpendicular to AB; AD, as was shewn (Prop. XXX.) will represent the force with which the body descends down the inclined plane AB. And, since the spaces through which a body falls in any given time must be as the forces which move them, the space through which the body will fall down the inclined plane AB, is to that through which it will fall perpendicularly in the same time, as the force AD, to the force AC. But AD is to AC (El. VI. 8. Cor.) as AC the elevation to AB the length of the plane; therefore the space through which the body will fall in a given time down the inclined plane AB, will be to the space through which it would fall perpendicularly in the same time, as the elevation of the plane to its length.

COR. A body would fall down the inclined plane from A to D, in the same time in which it would fall perpendicularly from A to C. For, the spaces passed through in any given time are as AC to AB, that is, (El. VI. 8. Cor.) as AD to AC: consequently, if AC is the space passed through in any given time by the body falling freely, AD will be the space passed through in the same time, down the inclined plane AB.

### P R O P. XXXII.

The velocity acquired in any given time by a body descending down an inclined plane, is to the velocity acquired in the same time  
by



by a body falling freely, as the elevation of the plane to the length.

In an uniformly accelerated motion, the velocities produced in equal times are as the forces which produce them : but (by Prop. XXX.) the force with which a body descends down an inclined plane, is to that of its perpendicular descent, as the height of the plane to its length : therefore the velocities produced in equal times are in the same ratio.

P R O P. XXXIII.

The time in which a body moves down an inclined plane, is to that in which it would fall perpendicularly from the same height, as the length of the plane to its elevation.

The square of the time in which AB is passed over, is to the square of the time in which AD is passed over (compare Prop. XXVI. with Prop. XXIX. Cor.) as AB to AD, that is, since AB, AC, AD (El. VI. 8. Cor.) are continued proportionals, as the square of AB to the square of AC. Therefore the times themselves are as the lines AB, AC, that is, as the length of the plane to its elevation.

Plate 1.  
Fig. 11.

COR. Hence if several inclined planes have equal altitudes, the times in which those planes are described by bodies falling down them, are as the lengths of the planes. For the time of the descent down AC is to the time of the fall down AB, as AC to AB ; and the time of the fall down AB is to the time of the descent down AG, as AB to AG ; therefore (El. V. 11.) the time of the descent from A to C is to the time of descent from A to G, as AC to AG, that is, the times are as the lengths of the planes.

Plate 1.  
Fig. 12.

P R O P. XXXIV.

A body acquires the same velocity in falling down an inclined plane, which it would acquire by falling freely through the perpendicular elevation of the plane.

The square of the velocity which a body acquires by falling to D, is (by Prop. XXVII. compared with Prop. XXIX. Cor.) to the square of the velocity it acquires by falling to B, as the space AD is to the space AB, that is, (El. VI. 8. Cor.) as the square of AD is to the square of AC ; and consequently the velocity at D is to the velocity at B, as AD is to AC. But, because AD and AC (Prop. XXXI. Cor.) are passed over in the same time, the velocity acquired at D is (by Prop. XXXII.) to that which is acquired at C, as AD to AC.

Plate 1.  
Fig. 11.

Since



Since then the velocity at D has the same ratio to the velocities at B, and at C, namely, the ratio of AD to AC, the velocities at B and C (El. V. 9.) are equal.

Plate 1.  
Fig. 12.

COR. Hence the velocities acquired by bodies falling down inclined planes are equal in equal times, where the heights of the planes are equal. The velocities acquired in falling from A to C, and from A to G, are each equal to the velocity acquired in falling from A to B, and therefore equal to one another.

### P R O P. XXXV.

A body falls perpendicularly through the diameter, and obliquely through any chord of a circle, in the same time.

Plate 1.  
Fig. 12.

In the circle ADB, let AB be a diameter, and AD any chord; draw BC a tangent to the circle at B; produce AD to C, and join DB. Because ADB (El. III. 31.) is a right angle, a body (by Prop. XXXI. Cor.) will fall from A to D on the inclined plane in the same time in which it will fall from A to B perpendicularly. In like manner let the chord AE be produced to G; and because AEB is a right angle, a body will fall from A to E in the inclined plane in the same time in which it would fall from A to B.

COR. Hence all the chords of a circle are described in equal times.

### P R O P. XXXVI.

If a body descends along several contiguous planes, the velocity which it acquires by the whole descent, is the same which it would acquire, if it fell from the same perpendicular height along one continued plane; and this velocity will be the same with that which would be acquired by the perpendicular fall from the elevation of the planes.

Plate 1.  
Fig. 13.

Let AB, BC, CD, be several contiguous planes; through the points A and D, draw HE, DF, parallel to the horizon, and produce the contiguous planes CB, CD, to G and E. By Prop. XXXIV. Cor. the same velocity is acquired at the point B, whether the body descends from A to B, or from G to B. Therefore, the line BC being the same in both cases, the velocity acquired at C must be the same, whether the body descends through AB, BC, or along GC. In like manner, it will have the same velocity at D, whether it falls through AB, BC, CD, or along ED, that is (by Prop. XXXIV.) its velocity will be equal to the velocity acquired by the perpendicular fall from H to D.

COR.



COR. Hence if a body descends along any arc of a circle, or any other curve, the velocity acquired at the end of the descent is equal to the velocity acquired by falling down the perpendicular height of the arc; for such a curve may be considered as consisting of indefinitely small right lines, representing contiguous inclined planes.

## P R O P. XXXVII.

If two bodies fall down two or more planes equally inclined, and proportional, the times of falling down these planes will be as the square roots of their lengths.

Let the inclined planes be AB, BC, DE, EF: let AG, DH, be lines drawn parallel to the horizon; let AB, DE, be equally inclined to the plane of the horizon, and also BC, EF; let AB be to DE as AG to DH, and as BC to EF, and draw GB, HE.

Plate 1.  
Fig. 14.

Because ABG, DEH, are similar triangles, AB is to DE (El. VI. 4.) as BG to EH, and  $\sqrt{AB}$  to  $\sqrt{DE}$  as  $\sqrt{BG}$  to  $\sqrt{EH}$ : also AB is to DE as  $AB+BC$  is to  $HE+EF$ , and  $\sqrt{AB}$  to  $\sqrt{DE}$  as  $\sqrt{AB+BC}$ , or  $\sqrt{GC}$ , is to  $\sqrt{HE+EF}$ , or  $\sqrt{HF}$ .

And since (by construction) AB is to DE as BC to EF, AB is to DE, as  $AB+BC$  is to  $DE+EF$ , and  $\sqrt{AB}$  to  $\sqrt{DE}$ , as  $\sqrt{AB+BC}$  to  $\sqrt{DE+EF}$ . But AB, DE, being planes equally inclined, the accelerating force of gravitation will be the same upon each, and the bodies descending upon them may be considered as falling down different parts of the same plane. Hence (Prop. XXVI. Cor. 1. and XXIX. Cor.) the time of descent along AB is to that along DE, as  $\sqrt{AB}$  to  $\sqrt{DE}$ ; and the time of descent along GC is to that along HF, as  $\sqrt{GC}$  is  $\sqrt{HF}$ , that is, as  $\sqrt{AB}$  to  $\sqrt{DE}$ . Again, the time of descent along GB is to that along HE, as  $\sqrt{BG}$  is  $\sqrt{EH}$ , that is, as  $\sqrt{AB}$  to  $\sqrt{DE}$ . Since, therefore, the time of descent along the whole plane GC is to that along the whole plane HF, as  $\sqrt{AB}$  to  $\sqrt{DE}$ , and that of the part GB is also to that of the part HE, as  $\sqrt{AB}$  to  $\sqrt{DE}$ , the time of descent along the remainder BC is to that along the remainder EF (El. V. 19.) as  $\sqrt{AB}$  to  $\sqrt{DE}$ . Consequently, the time of descent down  $AB+BC$  is to that down  $DE+EF$ , as  $\sqrt{AB}$  to  $\sqrt{DE}$ , that is, as  $\sqrt{AB+BC}$  to  $\sqrt{DE+EF}$ .

COR. Hence if bodies descend through arcs of circles, the times of describing similar arcs will be as the square roots of the arcs. For such similar arcs may be considered as composed of an equal number of proportional sides, or planes, having the same inclination to each other, and their elevations equal; whence by this proposition, the times of descent will be as the square roots of the lengths of the arcs.



## P R O P. XXXVIII.

If a body be thrown up along an inclined plane or the arc of a curve, it will, in the same time rise, to the same height, from which with equal force it would have descended; and any velocity will be lost in the same time in which it would, in descending, have been acquired.

For the same force of gravitation has in every respect the same efficacy to retard the motion of bodies ascending, as to accelerate them descending on an inclined plane or curve.

## S E C T. III.

*Of the Pendulum and Cycloid.*

DEF. V. A pendulum is a heavy body hanging by a cord or wire, and moveable with it upon a center.

## P R O P. XXXIX.

The vibrations of a pendulum are produced by the force of gravitation.

Plate 1.  
Fig. 15.

Let the ball A, suspended from the center B by the chord BA, be drawn up to C and let fall from thence: it will descend by the force of gravitation to A, from whence (being prevented from falling farther by the chord) it will proceed (by Prop. XXXVI. Cor.) with a velocity equal to that which it would have acquired in falling perpendicularly from E to A, which will carry it on the opposite side to the height from which it fell. Being brought back again towards A by the force of gravitation, it will acquire a new velocity which will carry it towards C: and in this manner it will vibrate by the force of gravitation, till the resistance of the air, and the friction of the string, stop its motion.

EXP. Let a pendulum vibrate freely.

PROP.



## P R O P. XL.

The same pendulum, vibrating in small unequal arcs, performs its vibrations nearly in equal times.

In the circle  $CGA$ , the small arcs  $CA$ ,  $EA$ , will differ little from their respective chords in length or declivity. But (by Prop. XXXVI. Cor.) the times in which the chords are passed over are equal; therefore the times of describing the arcs  $CA$ ,  $EA$ , and also (by Prop. XXXVIII.) of describing their doubles  $CAD$ ,  $EAF$ , will be nearly equal.

Plate 1.  
Fig. 16.  
and 17.

EXP. Two equal pendulums vibrating in small, but unequal arcs, will, for a long time, keep pace in their vibrations.

## P R O P. XLI.

If a pendulum vibrates through small arcs of circles of different lengths, the velocity it acquires at the lowest point, is as the chord of the arc which it describes in its descent.

Let  $BA$  be the pendulum, and  $CAD$ ,  $EAF$ , the arcs through which it vibrates; and draw the horizontal lines  $EK$ ,  $CH$ . The velocity acquired in falling from  $H$  to  $A$ , is (by Prop. XXVI. Cor.) to that acquired by falling from  $G$  to  $A$ , as  $\sqrt{HA}$  to  $\sqrt{GA}$ , that is, (by El. VI. 8. Cor.) as  $CA$  to  $GA$ . For the same reason, the velocity acquired in falling from  $G$  to  $A$ , is to that acquired in falling from  $K$  to  $A$ , as  $GA$  to  $EA$ . Consequently, *ex æquali*, the velocity acquired in falling from  $H$  to  $A$ , is to that acquired by falling from  $K$  to  $A$ , as  $CA$  to  $EA$ . But (by Prop. XXXVI. Cor.) the velocity acquired in falling from  $H$  to  $A$  is equal to that from  $C$  to  $A$ ; and the velocity acquired in falling from  $K$  to  $A$  is equal to that from  $E$  to  $A$ . Therefore the velocity acquired in descending through the arc  $CA$ , is to that through  $EA$ , as the chord  $CA$  to the chord  $EA$ : and the same may be shewn concerning the remaining half of the vibrations,  $AF$ ,  $AD$ .

Plate 1.  
Fig. 16.  
and 17.

COR. Hence the lengths of the chords of arcs through which pendulums move, are measures of velocity.

Fig. 17.

## P R O P. XLII.

The time of the descent and ascent of a pendulum, supposing it to vibrate in the chord of a circle, is equal to the time in which a  
F 2 body



body falling freely would descend through eight times the length of the pendulum.

For the time of the descent of a body upon the chord, is (by Prop. XXXVI.) equal to that of the fall through the diameter of the circle, which is twice the length of the pendulum: but in double that time, that is, in the descent and ascent, or whole vibration, the body would fall (by Prop. XXVII.) through four times the space, that is, through eight times the length of the pendulum.

### P R O P. XLIII.

The times in which pendulums of different lengths perform their vibrations, are as the square roots of their lengths.

Plate 1.  
Fig. 18.

Let the two pendulums, AB, CD, be of different lengths. The time in which the first, AB, vibrates through a chord, is equal to that in which a body (Prop. XXXV.) would fall freely through twice AB, the diameter of the circle of which AB is radius: in like manner, the time in which CD vibrates, is equal to that in which a body would fall through twice CD. But the times in which a body would fall through these different spaces are (Prop. XXVI. Cor. 1.) as the square roots of the spaces, that is, as the square roots of AB and CD, the lengths of the pendulums: therefore the vibrations are in the same ratio.

Fig. 19.

COR. The times in which pendulums of unequal lengths vibrate, are as the square roots of the similar arcs through which they move. If BA, BC, be pendulums of different lengths vibrating in the similar arcs FG, DE. Since the times of vibration are as the square roots of the lengths BA, BC, and that similar arcs are as the diameters, the times of vibration, are as the square roots of the arcs, FA, DC, or of their doubles, FG, DE.

EXP. Two pendulums, the lengths of which are as 1 to 4, will perform their vibrations in times as 1 to 2, that is, the shorter pendulum will make two vibrations, whilst the longer makes one.

### P R O P. XLIV.

The squares of the times in which a pendulum of a given length performs its vibrations, are inversely as the accelerating forces, or gravities.

By Prop. XXVII. where the accelerating force is given, the space described is as the square of the time in which it is described. And since, in any given moving body, the  
velocity



velocity must be as the accelerating force, where the square of the time, or the time itself, is given (by Prop. II.) the space described will be as the accelerating force. Consequently, where neither the accelerating force, nor the square of the time, is given, the space described will be in the ratio compounded of both. If then the space described be called  $S$ , the accelerating force  $A$ , and the square of the time  $T^2$ ,  $S$  will be as  $T^2 A$ , whence  $\frac{T^2 A}{A}$ , or  $T^2$  is as  $\frac{S}{A}$ . But, when the spaces are equal,  $S$  is a given quantity: whence (since fractions whose numerators are given, are inversely as their denominators)  $\frac{S}{A}$  is inversely as  $A$ . But  $T^2$  is as  $\frac{S}{A}$ ; therefore where  $S$  is given  $T^2$  is inversely as  $A$ ; that is, where the spaces described are equal, the squares of the times in which they are described are inversely as the accelerating forces. And if the squares of the times of falling bodies are inversely as their accelerating forces, the squares of the times in which pendulums vibrate, are in the same ratio, on account of the constant equality between the time of vibration and that of the descent through eight times the length of the pendulum, by Prop. XLII.

COR. Hence, if the same pendulum, at different parts of the earth, performs its vibrations in different times, the forces of gravitation will, in those places, be inversely as the squares of those times.

## L E M M A I.

*If, from X as the center, with any distance XA, a quadrant of a circle ADB be described, and in the right line AX a body descends with such force, that its velocity in any points M, N, &c. shall be always as MD, NP, &c. the sines of the arcs AD, AP: the time in which the body will descend from A to X, will be equal to the time in which it would describe the whole arc ADB with the uniform velocity, expressed by XB, acquired by the falling body when it arrives at X: also, the time of the fall through any space AM, will be to the time of the fall through any other space AO, as the arc AD to the arc AQ; and the force with which the body is accelerated in any place M, will be as MX the distance of that place from the center.*

Plate 2.  
Fig. 1.

Let DP be a part of the circumference taken indefinitely small, and therefore not assignably differing from a right line; join DX; and draw DL perpendicular to NP. Because the triangles MDX, LDP, are similar (having each a right angle, and the angles MDX, LDP, whose common complement is LDX, equal) MD will be to DX as LD

or



or  $MN$  to  $DP$ . But, by the hypothesis,  $MD$  is as the velocity of the descending body at the point  $M$ , that is, as the velocity with which the indefinitely small line  $MN$  is described; and  $XD$  is as the velocity last acquired by the falling body at  $X$ , that is, as the uniform velocity with which the arc  $DP$  is described. The velocity therefore of the body descending through the indefinitely small line  $MN$ , will be to the velocity of the body moving along the arc  $DP$ , as  $MN$  to  $DP$ . Wherefore, since the velocities are proportional to the spaces passed over, the times wherein those spaces,  $MN$ ,  $DP$ , are described will be equal. After the same manner it may be proved, that any other indefinitely small portion of the circumference,  $PQ$ , may be described with the velocity  $XB$ , in the same time in which the corresponding line  $NO$  will be described with the corresponding velocity  $NP$ : and consequently, by composition, the falling body will descend through all the indefinitely small portions of the perpendicular  $AX$ , that is, through the whole line, in the time in which all the corresponding parts of the circumference, that is, the whole quadrant  $ADB$ , is described with an uniform velocity as  $XB$ .

Moreover, the time in which the falling body descends from  $A$  to  $M$ , is equal to the time in which the arc  $AD$  is passed over; and the time in which it descends from  $A$  to  $O$  is equal to the time in which the arc  $AQ$  is described: but the time in which the arc  $AD$  is passed over, is to that in which the arc  $AQ$  is passed over (since they are both described with the same velocity) as the arc  $AD$  to the arc  $AQ$ : therefore the time of descent from  $A$  to  $M$ , will be to the time of descent from  $A$  to  $O$ , as the arc  $AD$  to the arc  $AQ$ : and consequently, by division, the time of descent through  $AM$  will be to the time of descent through  $MO$ , as the arc  $AD$  to the arc  $DQ$ .

Lastly, let the arcs  $DP$ ,  $PQ$ , be equal; join  $XP$ , and from  $P$  let fall  $PS$  perpendicular to  $OQ$ : the time of descent through  $MN$  will be equal to that through  $NO$ : and, since the triangles  $LDP$ ,  $MDX$ , are similar, and also  $SPQ$ ,  $NPX$ ;  $LP$  will be to  $DP$  or  $PQ$ , as  $MX$  to  $XD$  or  $XP$ ; also  $PQ$  is to  $SQ$  as  $XP$  to  $XN$ ; and consequently, (El. V. 11.)  $LP$  will be to  $SQ$  as  $XM$  to  $XN$ . But  $LP$  is as the increment of the velocity acquired while the body is passing over  $MN$ , and  $SQ$  is as the increment of the velocity acquired in passing over in an equal time the indefinitely small line  $NO$ ; and the forces with which the body is accelerated at  $M$  and  $N$ , are as the increments of the velocities generated in equal times: the accelerating forces at  $M$  and  $N$  will therefore be as the lines  $LP$ ,  $SQ$ ; that is, the force with which the body is impelled at  $M$  is to that at  $N$ , as the distance  $XM$  to the distance  $XN$ , or, the accelerating forces are as their distances from the center.

COR. Hence, conversely, if a body, descending from  $A$  to  $X$  is impelled by a force which is as its distance from the center  $X$ , and that force at the beginning of the motion is expressed by the right line  $CE$  (the arc  $AE$  being taken indefinitely small) the velocities of the same body in any places  $M$ ,  $O$ , will be expressed by the fines  $MD$ ,  $OQ$ ; and the times by the arcs  $AD$ ,  $AQ$ ; and the increments of the velocities, or, if the arcs increase equally, the accelerating forces, will be expressed by the increments of the fines.

L E M.



## L E M. II.

*If a body, moving along the line AX, be impelled by forces proportional to its distance from the point X; from whatever height it falls, it will arrive at the point X in the same time; and this time will be to the time in which it would move over the whole line AX with the velocity which it acquires by falling from A to X, as half the circumference of a circle to its diameter.*

Plate 2.  
Fig. 2.

Let two bodies be let fall from the points A and P at the same time; and let them be impelled by forces proportional to their distances from the point X; these bodies will come to X at the same time. From X as a center, with the radii XA and XP, describe the two quadrants AB and PQ; and let the force by which the body A is impelled, or, which is the same thing, its velocity at the beginning of motion, be represented by RS, the sine of the indefinitely small arc AS: it is manifest (from the Cor. of the preceding Lemma) that its velocity, after the fall to X will be properly expressed by XB. But, by hypothesis, the force by which the body at A is accelerated, is to that by which the body at P is accelerated, as AX is to PX, that is (since the arcs AS and PN are similar) as RS to MN. As therefore RS expresses the first velocity of the body moving from A, MN will express the first velocity of the body moving from P: and consequently (by the Cor. to the last Lemma) XQ will express the velocity of the body moving from P, when it arrives at X. Farther, the time of the fall from A to X (by Lemma I.) is equal to the time in which the arc AB would be described with a velocity as XB: and the time of the fall from P to X is equal to the time in which the arc PQ would be described with a velocity as XQ. But (because the line XQ is to the line XB as the arc PQ to the arc AB, and that the spaces passed over are proportional to the times) the time in which the arc AB is described with the velocity XB is equal to the time in which the arc PQ is described with the velocity XQ. Wherefore the time of the fall from A to X will be equal to the time of the fall from P to X.

Again, since (by Lem. I.) the time in which a body would fall from A to X is equal to the time in which it would move over the arc AB, with its last acquired velocity at X; and since it is evident, that the time in which a body would move over the arc AB with the velocity at X is to the time in which it would move over AX with the same velocity, as AB is to AX; the time in which a body would fall from A to X is to the time in which it would move over AX with the last acquired velocity, as AB to AX. But AB is to AX, as twice AB to twice AX, that is, as half the circumference of a circle is to its diameter. Therefore the time in which a body would fall from A to X, is to the time in which it would move over AX with its last acquired velocity, as half the circumference of a circle is to its diameter.

DEF.



Plate 2.  
Fig. 3.

DEF. VI. If a circle, as FCH, be rolled along the line AB, till it has turned once round; the point C in its circumference, which at first touched the line at A, will describe the curve line ACXB, which curve is called a *Cycloid*. The right line AB is its *base*: the middle point X is its *vertex*: a perpendicular, as XD, let fall from thence to the base, is its *axis*: and the circle FCH, or any other as XGD, equal thereto, is called the *generating circle*.

### L E M. III.

*If on XD, the axis of the cycloid, as a diameter, the generating circle XGD be described; and if from a point in the cycloid, as C, the line CIK be drawn parallel to the base, the portion of it CG, will be equal to the arc GX.*

Because the generating circles FCH, DGX, are equal, (the diameter HF being drawn) KG is equal to CI; whence, adding GI to both, KI will be equal to CG: and KI, by the construction, is equal to DF; therefore CG is equal to FD. By the description of the cycloid, the arc CF is equal to the line AF: and by the construction the arc CF is equal to DG: therefore AF is equal to DG: but, by the description of the cycloid, AFD is equal to DGX; consequently, FD is equal to GX: and CG was proved to be equal to FD: therefore CG is equal to GX.

### L E M. IV.

*A tangent to the cycloid at the point C is parallel to GX a chord of the circle DGX.*

Draw  $ck$ , parallel to the base and indefinitely near to CK meeting the cycloid in  $c$ , the axis in  $k$ , and the circle in  $g$ . Let  $Cu$  and  $Gn$ , parallel to the axis, meet  $ck$  in  $u$  and  $n$ , and from T, the center of the circle XGDM, draw the radius TG. Since  $cg$  is equal (Lem. III.) to  $gX$ ,  $gk$  being added to both,  $ck$  will be equal to  $Xg + gk$ : therefore  $cu$  the excess of  $ck$  above CK is equal to  $Gg + gn$ , the excess of  $Xg + gk$  above  $XG + GK$ . And, if we suppose  $ck$  to approach towards CK, as  $Gg$  and  $gn$  vanish, the triangles  $Ggn$  and  $GTK$  become similar; for the angle  $Ggn$  is then equal to the angle  $TGK$ , since both have the same angle  $nGT$ , or its alternate  $GTK$ , as their complement. Whence  $Gg$  is to  $gn$  as  $TG$  to  $TK$ , and (El. V. 18.)  $Gg + gn$  to  $gn$ , as  $TG + TK$  or  $DK$  to  $TK$ ; but  $Gn$  is to  $gn$  as  $GK$  to  $TK$ ; therefore  $Gg + gn$  is to  $Gn$  as  $DK$  is to  $GK$ , that is (El. VI. 8.) as  $GK$  to  $XK$ . And consequently  $cu$  (shown to be equal to  $Gg + gn$ ) is to  $Gn$ , or  $Cu$ , as  $GK$  to  $XK$ : and if the chord  $Cc$  be drawn, the triangles  $Cuc$ ,  $XKG$ , will be similar: so that the chord  $Cc$  (as the points C and  $c$  coincide) becomes parallel to  $XG$ ; therefore the tangent of the cycloid at C is parallel to  $KG$ .

L E M.



## L E M. V.

*If from a point of the cycloid, as L, the line LMK be drawn parallel to the base AB, the arc XL of the cycloid, will be double of XM the chord of the circle corresponding thereto.*

Plate 2.  
Fig. 3.

Draw the line  $Sk$  parallel and indefinitely near to  $LK$  crossing the circle in  $R$ , and the chord  $XM$  produced, in  $P$ : join the points  $X$  and  $R$ ; on  $MP$  let fall the perpendicular  $RO$ ; and draw  $MN$ ,  $XN$ , tangents to the circle at  $M$  and  $X$ . Then will the lines  $XN$  and  $kS$ , being each perpendicular to the diameter  $DX$ , be parallel; and the triangles  $MNX$ ,  $MPR$ , having their angles at  $M$  vertical, and at  $P$  and  $X$  alternate, will be similar. But the tangents  $NX$  and  $NM$  are equal (El. III. 36.) whence the lines  $PR$  and  $RM$  are also equal: the triangle  $RMP$  is therefore isosceles; and  $RO$  being perpendicular to its base  $MP$ ,  $MO$  (El. I. 26.) is equal to  $OP$ ; whence  $MP$  is equal to twice  $MO$ . The indefinitely small arc  $LS$  of the cycloid will not assignably differ from a portion of a tangent drawn through the point  $L$ .  $LS$  may therefore (Lem. IV.) be said to be parallel to  $MP$ , and consequently (from the parallelism of  $ML$  and  $PS$ ) equal to it: it is therefore equal also to twice  $MO$ . But  $LS$  is the difference between the cycloidal arcs  $XL$  and  $XS$ ; and  $MO$  is the difference between the chords  $XM$  and  $XR$ : for since  $XO$  and  $XR$  are indefinitely near to each other,  $RO$  which is perpendicular to one of them, may be considered as perpendicular to both: the indefinitely small difference therefore between any two arcs of the cycloid is twice that which is between the two corresponding chords of the circle; and the same is true when the magnitude of the difference is assignable, because such difference is compounded of indefinitely small parts. Now, any arc whatsoever may be considered as a difference between two arcs, and consequently any arc, as  $XL$ , is double of the corresponding chord  $XM$ .

COR. Since when the arc  $XL$  becomes  $XB$ , the corresponding chord  $XM$  becomes  $XD$  the diameter of the circle  $DMX$ ; it is obvious, that the femicycloid  $BX$ , or  $AX$ , is equal to twice  $DX$  the diameter of the generating circle  $DMX$ .

## L E M. VI.

*If a body descends in a cycloid, the force of gravity, so far as it acts upon the body in causing it to descend along the cycloid, will be proportional to the distance of the body from the lowest point of the cycloid.*

Let the cycloid be  $AXB$ , whose base is  $AB$ , and its axis  $DX$ ; on which last, as a diameter, describe the generating circle  $DQX$ : draw the chords  $OX$  and  $QX$ ; through the points  $O$  and  $Q$ , and parallel to the axis  $AB$ , draw the lines  $LS$  and  $MR$ ; draw also the tangents  $LV$  and  $MY$ . Then because (by Lem. IV.) the tangent  $LV$  is parallel to  $OX$ , and the tangent  $MY$  parallel to  $QX$ , it is obvious that gravity exerts the same power upon a body descending in the cycloid at  $L$  (because it then descends in the tangent  $LV$ ) as it would do upon the same body descending along the chord  $OX$ : and for the like

Plate 2.  
Fig. 4.



reason, it exerts the same force upon it when it comes to M, that it would do if it were descending along QX: but (from Prop. XXXV.) the power or force of gravity upon bodies descending along the chords OX and QX, are as the lengths of those chords; that is, by Lem. V. (halves being proportional to their wholes) as the length of the cycloidal arcs LX and MX. The force therefore of gravity upon a body descending in the cycloid at the point L is to its force upon the same when at M (as may be said of any other corresponding points) as the space or distance it has to move over in the former case, before it reaches the lowest point X, to that which it has to pass over in the latter, before it arrives at the same point.

## P R O P. XLV.

If a pendulum be made to vibrate in a cycloid, all its vibrations, however unequal in length, will be performed in equal times.

The force of gravity (by Lem. VI.) so far as it causes a body to descend in a cycloid, is proportional to the distance of that body from the lowest point: imagine then that body to be a pendulum vibrating in the cycloid, and from whatever point it sets out, it will (by Lem. II.) come to the lowest point in the same time: and consequently, since the same may be easily inferred in its ascending from that point, all its vibrations, be they large or small, will be performed in the same time.

## P R O P. XLVI.

To make a pendulum vibrate in a given cycloid.

Plate 2.  
Fig. 4.

Let AXB be the given cycloid; its base AB, its axis DX, and its generating circle DQX, as before: produce XD to C, till DC is equal to DX: through C draw the line EF parallel to AB, and take CE and CF, each equal to AD or DB; and on the line CE as a base, and with the generating circle AGE equal to DQX, describe the semicycloid CTA, whose vertex will therefore touch the base of the given cycloid in A. And on the line CF also as a base, describe an equal semicycloid CB. Let the semicycloids CA, CB, represent thin plates of metal bent to their figure, and on the point C, hang the pendulum CTP by a flexible line equal in length to the line CX. The upper part of its string (as CT, in its present situation in the figure) as it vibrates, will then apply itself to the cycloidal cheeks CA and CB, and a ball at P will oscillate in the given cycloid AXB.

Draw TG and PH each parallel to the base AB, and draw AG, and DH. Then (Lem. V. Cor.) AC is equal to twice AE; and by construction, twice DC, that is, twice AE, is equal to CX; therefore AC is equal to CX. Also, by construction, CTP is equal to CX, that is, to ATC: whence, taking away CT, AT is equal to TP. By Lem. IV. GA is parallel to TP; and, by construction, AK is parallel to GT; therefore GA is equal to TK, and GT to AK; but (Lem. V.) GA is half TA; therefore TK is equal to half TA: since therefore it has been proved that TA is equal to TP, TK is equal to half TP, that is, to KP. Hence it is manifest, that the parallel lines GT, PH, are equally distant



distant from AD, the arc GA equal to the arc DH, the chords GA and DH parallel, and GE equal to HX. And, because GA has been shewn to be parallel to TK, and also to DH, KP and DH are parallel; whence KD is equal to PH. But (Lem. III.) GT, that is, AK is equal to the arc AG: and by the description of the semicycloid CTA, AKD is equal to AGE; therefore KD is equal to EG, that is, PH is equal to HX. And (by Lem. III.) if PH be equal to HX, P is a point in the cycloid AXB. The ball of the pendulum therefore being at that point, is in the given cycloid.

SCHOL. It is easy to conceive, that in a pendulum there must be some one point, on each side of which the momenta of the several parts of the pendulum will be equal, or in which the whole gravity of the pendulum might be collected without altering the time of its vibrations. This point, which is called the *center of oscillation*, is different from the center of gravity: for if a plane, perpendicular to the string of the pendulum AB be conceived to pass through the center of the ball B, bisecting it; the velocity of the lower half, and consequently its momentum, will, in vibration, be greater than that of the upper half: consequently the center of oscillation must be farther from A than the center of gravity is; and a plane passing through the center of oscillation will divide the ball into two unequal parts, so that the greater quantity of matter above it, shall compensate for the greater velocity below it, and the momenta on each side be equal. If the pendulum be an inflexible rod every where of equal size, it is found, that the distance of the center of oscillation from the point of suspension is two thirds of the length of the rod.

Plate 1.  
Fig. 18.

If, whilst a pendulum is in motion, it meets with an obstacle at its center of oscillation sufficient to stop it, the whole motion of the pendulum will cease at once, without any jarring: for the obstacle resists equal momenta above and below this point; which is therefore also called the *center of percussion*.

EXP. Let an iron rod, equal in size throughout, moving as a pendulum, be stopped by an obstacle meeting it  $\frac{2}{3}$  of its length from the point of suspension.

## S E C T. IV.

*Of the Center of Gravity.*

## P R O P. XLVII.

In every body there is a *center of gravity*, or a point about which all its parts balance each other.

Plate 2.  
Fig. 5.

Let AB be an inflexible rod, throughout uniform and of the same density: let it be supported at the point C, equally distant from its extreme points A and B, by the prop C. Let A and B be indefinitely small and equal portions of the rod AB. These portions, A and B, tend towards the center of the earth with equal forces of gravitation. They would likewise, without obstruction, move with equal velocities: for, if the rod AB be moved on its prop till it comes into the position DE, the velocities of the parts A, B, or F, will be as the spaces over which they pass in the same time; that is, as the arcs EB, AD, or FG; which arcs are as their respective circumferences, or as their diameters or *radii*: whence the velocity of the part B is to the velocity of the part A, or F, as BC is to AC, or FC. And the quantities of matter in A and B are by supposition equal. Therefore, if the parts A and B were in motion, they would have equal momenta; that is, the efforts which A and B make to descend towards the earth, are equal. But these efforts counteract each other: for, whilst the portion A endeavours with a certain force to draw down one arm of the rod, the other portion B endeavours with the same force to draw down the other arm, that is, since the rod is inflexible, to raise the portion A. Therefore the portion A is acted upon by two equal forces in contrary directions, and consequently must be at rest. For the same reason, the portion B will be at rest. And the same may be shewn concerning any other equal portions, at equal distances from C, in the rod AB. Therefore the rod will be at rest; that is, the parts on each side of the point will balance each other, and C will be the *center of gravity*.

If the rod were placed oblique to the prop C, indefinitely small and equal parts being taken, as before, at equal distances from C, and resolving each oblique force into an horizontal and perpendicular force (as in Prop. XVI.) it might be shewn, by a similar manner of reasoning, that they would tend towards the earth with equal forces, and consequently, that an equilibrium would be produced.

And if, instead of equal portions of the rod, portions of matter were placed at different distances, which should be to each other inversely as those distances, as at F and B, the equilibrium



equilibrium would still be preserved : for the forces with which such portions of matter, so situated, would endeavour to descend, would be equal, when the quantities of matter, multiplied into the velocities with which they are endeavouring to move, that is, into their distances (Prop. XI. Cor.) are equal ; as will be more fully shewn, in treating of the Mechanical Powers.

Since, therefore, all the parts of any irregular body may be referred to some one of the above cases, it is manifest, that there is in every body a certain point, the parts on each side of which, balance each other.

## P R O P. XLVIII.

If the center of gravity in any body be supported, the whole body is supported ; if this center be not supported, the body will fall.

For, when the center of gravity is supported, the body rests on a prop on which the parts on each side, acting with equal force against each other, will (Prop. XLVII.) be in equilibrio, and neither side will move ; but when this center is not supported, but the body has a prop under some other point, the parts of the body on one side of that other point will over-balance the parts on the other side, and the body will fall.

COR. Whenever a body moves by the power of gravitation, or falls, its center of gravity descends : for if this center do not descend, it must be supported ; and if the center be supported, the whole body is sustained or kept from falling.

EXP. 1. Let a board of a circular form be sustained perpendicularly on its center of gravity, it will be at rest in any position.

2. A beam turning on an axis which passes through its center of gravity, will rest in the same manner.

3. A beam, whose axis passes through a point which is directly above the center of gravity, will be at rest only when the beam is parallel to the plane of the horizon, because the center of gravity will be then fallen as low as possible.

4. A cylinder, which has its center of gravity near one of its sides, will roll up an inclined plane, if the side nearest the center of gravity be placed towards the upper part of the plane ; for this center, endeavouring to descend, will carry the cylinder forward in the ascending direction of the plane.

5. Let a body consisting of two equal and similar cones united at their bases, be placed upon the edges of two straight and smooth rods, which at one end meet in an angle, and rest upon an horizontal plane, and at the other are raised a little above the plane, the body will roll towards the elevated end of the rulers, and appear to ascend, while its center of gravity descends ; as may be seen by applying a string horizontally above the path of the base of the cones.

P R O P.



## P R O P. XLIX.

If the line of direction comes within the base on which any body is placed horizontally, the body will be sustained; otherwise it will fall.

Plate 2.  
Fig. 7.

In the body ABDE let C be the center of gravity. The line of direction CO (that is, the line drawn from the center of gravity towards the center of the earth) being within the base DE, the body will be supported, because the weight presses upon the base. Also since the body cannot fall towards K without turning round on the point E, the point C must in the motion ascend towards F, contrary to Prop. XLVIII. Cor. But in the position of the body *abde*, *co* the line of direction falling out of the base, *c* in its motion towards *k* descends, and the body will fall.

EXP. I. Any body of a cylindrical or other regular form, so placed upon its base, that its line of direction does not come within the base (which may be seen by a cord and weight suspended from the center of gravity) will fall; otherwise it will not fall.

2. Let two bodies be laid upon an inclined plane, the one a cube, the other a figure with many sides, and let the line of direction of the former fall within the base, and that of the latter without the base, the former body will *slide*, the latter *roll* down the plane.

DEF. VII. The center of motion is the point about which a body moves.

## P R O P. L.

A heavy body suspended on a center of motion will be at rest, if the center of gravity is directly under, or above, the center of motion; otherwise it will move.

Plate 2.  
Fig. 8.

If a heavy body E, hangs by a string on a center of motion C, the action of gravitation at E, is in the direction EL, contrary to the direction in which the string acts to prevent the body from falling. In this position, therefore, the opposite forces being equal and in contrary directions, destroy each other, and the body is at rest. But if the body is at *p*, one of the forces acts in the direction *pC*, and the other in the direction *pL*, that is, in directions oblique to each other, whence the body will move in the diagonal of the parallelogram formed by *pC*, *pL*. And in all cases, since (without the aid of mechanical powers afterwards explained) the force which sustains any body must be equal to its weight, the center of gravity can only be at rest when these forces are in the same line of direction,  
that



that is, when the center of gravity is directly under, or directly above, the center of motion.

EXP. A circular board sustained at a point above or below the center of gravity, will only be at rest when the center of gravity is at the lowest point, that is, in the line of direction; or when the center of gravity is in the same line above the center of motion.

SCHOL. If two or more bodies be united, they may be considered as one, and have a common center of gravity.

EXP. I. Let two unequal balls be fixed upon the ends of a wire, they will have a common center of gravity.

2. Lay a piece of wood, of any form, on a plane surface, so near the edge as just to be kept from falling: then fix a fork obliquely into each side of the piece of wood, in such manner as to cast the center of gravity of the whole mass farther from the edge of the plane on which the piece of wood rested, the whole will be supported.

3. In like manner an image may be made to imitate the motion of a sawyer.

4. A board, which of itself would fall from a table (its center of gravity lying beyond the edge of the table) may be made, in the same position, to support a vessel of water, hanging upon it *near the table*; if a stick, fixed with one end at the bottom of the vessel, and the other in a hole in the horizontal board, be long enough to push the vessel a little out of the perpendicular, that is, to bring the center of gravity of the whole under the table.

SCHOL. The common center of gravity of any number of bodies may be thus found. Let C be the common center of gravity of two bodies, dividing their distances (see Prop. XLVII.) in such manner, that AC is to CB, as B to A; whence  $A \times AC = B \times BC$ , and consequently, if the point C is supported, the bodies A and B balance each other. Suppose a third body, equal to the sum of A and B, placed in their common center of gravity C; from the point C draw a right line to the center of a third body D, which divide in O, so that OD may be to OC, as  $A+B$  is to D; then is O the common center of the three bodies, A, B, D. In the same manner may be found the common center of any number of bodies.

Plate 2.  
Fig. a.

### P R O P. LI.

If any number of bodies move uniformly in right lines, whether in the same or different directions, their common center of gravity is either at rest, or moves uniformly in a right line.

If two bodies, A and B, move towards each other in the same right line, having their common center of gravity C, and their momenta equal, the velocity of A will be to that of B, as the body B to the body A, that is (as was shewn Prop. XLVII.) as AC

Plate 2.  
Fig. 5.

to



to BC. Whence (Prop. VI.) whilst A passes through AC, B will pass through BC, and the bodies will meet in C, which is their center of gravity during their motion, and at the time of concurrence: therefore the point C remains at rest.

In the same manner, it may be shewn, that if the bodies recede from each other with uniform motions, the centers of gravity will be at rest.

*Plate 2.* Next, suppose that two bodies, A and B, move in different directions AC, BD, describing equal spaces AC, CE, and BD, DF, in equal times; their common center of gravity L, will move uniformly in a right line. Produce CA, DB, till they meet in G: make AG to GH, as AC is to BD; draw the right line AH; and through C and E drawn CI, EK, parallel to AH. AC is to HI (El. VI. 2.) as AG to GH, that is, as AC to BD: therefore (El. V. 9.) HI is equal to BD, and adding IB to each, HB is equal to ID. In like manner, CE is to IK, as AG to GH, that is, as AC to BD, or CE to DF: therefore (El. V. 9.) IK is equal to DF, and, adding KD to each, ID is equal to KF: but ID was proved to be equal to HB; therefore, KF is equal to HB. From L, the common center of gravity of the bodies A and B, draw LM parallel to BD: draw GM, and produce it till it cut CI, EK, in the points N and O; and through these points draw NP, OQ, parallel to BD. AL is to LB (El. VI. 2.) as AM to MH; and CP to PD, (as CN to NI, that is) as AM to MH; therefore (El. V. 11.) CP is to PD, as AL to LB, that is (because L is the common center) as B to A. Consequently, P will be the common center of the bodies when they are found in C and D: and in like manner, it may be shewn, that Q will be their common center, when they are in E and F. But, since ML is to HB, as AM to AH, that is, as CN to CI, that is, as NP to ID, and that HB has been proved to be equal to ID, ML is equal to NP; and, in like manner, NP equal to OQ: whence (ML, NP, OQ, being parallel to one another) the line LPQ is equal to the line MNO, and, the points P, Q, (any points of the line in which the common center of gravity is found as the bodies are moving from A to E, and from B to F) will be in a right line. Moreover, since (El. VI. 2.) AC is to CE, as MN to NO, or LP to PQ, and that AC is equal to CE, LP will be equal to PQ. Therefore the common center of gravity of the bodies A and B, is always in the same right line, and moves uniformly, or passes over equal spaces in equal times.

In like manner, the common center of these two bodies and any third body, or of the three bodies and a fourth, &c. being found, it may be proved that it moves uniformly in a right line.

COR. Hence it is manifest, that any forces acting upon a system of bodies, must affect the motion of the common center of gravity of that system, in the same manner as if the same force were similarly applied to a body equal to the sum of all the bodies, placed in the common center of gravity. And the mutual actions of the parts of a system upon each other, producing (by Prop. III.) equal momenta in contrary directions, cannot change the state of motion or rest of their common center of gravity. Consequently, the law



law of a system of bodies, as to motion or rest, is the same as that of one body, and is rightly estimated from the motion of its center of gravity.

## C H A P. VI.

*Of Motion as directed by certain Instruments called MECHANICAL POWERS.*

DEF. VIII. That body which communicates motion to another, is called *the Power*.

DEF. IX. That body which receives motion from another, is called *the Weight*.

DEF. X. The *Lever* is a bar moveable about a fixed point, called its *fulcrum*, or prop. It is in theory considered as an inflexible line without weight. It is of three kinds: the first, when the prop is between the weight and the power; the second, when the weight is between the prop and the power; the third, when the power is between the prop and the weight.

EXP. Let the three kinds of the lever be shewn, as in Plate 2, Fig. 12, 13, 14.

## P R O P. LII.

A power and weight acting upon the arms of a lever will balance each other, when the distance of the point at which the power is applied to the lever from the prop, is to the distance of the point at which the weight is applied, as the weight is to the power.

Let C be the center of motion in the lever KL; let A and B be any two powers applied to it at K and L, acting in the directions KA and LB. From the center of motion C, let CM and CN be perpendicular to those directions in M and N; suppose CM to be less

H

than

Plate 2.  
Fig. 11.



than  $CN$ , and from the center  $C$ , at the distance  $CN$ , describe the circle  $NHD$ , meeting  $KA$  in  $D$ . Let the power  $A$  be represented by  $DA$ , and let it be resolved into the power  $DG$  acting in the direction  $CD$ , and the power  $DF$  perpendicular to  $CD$ , by completing the parallelogram  $AFDG$ . The power  $DG$ , acting in the direction  $CD$  from the center of the circle, or wheel,  $DHN$  towards its circumference, has no effect in turning it round the center, from  $D$  towards  $H$ , and tends only to carry it off from that center. It is the part  $DF$  only that endeavours to move the wheel from  $D$  towards  $H$  and  $N$ , and is wholly employed in this effort. The power  $B$  may be conceived to be applied at  $N$  as well as at  $L$ , and to be wholly employed in endeavouring to turn the wheel the contrary way, from  $N$  towards  $H$  and  $D$ . If therefore the power  $B$  be equal to that part of  $A$  which is represented by  $DF$ , these efforts being equal and opposite, must destroy each other's effect; that is, when the power  $B$  is to the power  $A$ , as  $DF$  to  $DA$ , or, (because of the similitude of the triangles  $AFD$ ,  $DMC$ ) as  $CM$  to  $CD$ , or as  $CM$ , to  $CN$ , then the powers must be in equilibrio; and those powers will sustain each other, which are inversely as the distances of their directions from the center of motion.

Plate 2.  
Fig. 10.

Otherwise; Case 1. When the power acts perpendicularly: Let  $AB$  be the lever,  $C$  the prop,  $P$  the power,  $W$  the weight. The force with which any body moves being as its momentum (Prop. XIII.) and its momentum as the quantity of matter multiplied into the velocity (by Prop. XI. Cor.) the force with which the weight  $W$  would move in the first instant of its motion, if no other body counteracted it, would be as its quantity of matter multiplied into its velocity. But because the weight  $W$  is suspended from the lever  $AB$  at the point  $B$ , it would move with the same velocity as this point; which (as was shewn in Prop. XLVII.) is as the distance of the point  $B$  from the prop  $C$ , or  $D$ . The force therefore with which the weight  $W$  would move without any counteracting force, is as its quantity of matter multiplied into the distance of the point of suspension  $B$  from the prop  $C$ , or  $D$ . But the weight will be prevented from descending, if a force equal to that with which it would descend without obstruction, acts upon it in the contrary direction; that is, if a force be applied to raise the point  $B$  of the lever  $AB$ , equal to that with which the weight  $W$  would draw it downwards. Let the power  $P$  be suspended from the other extremity of the lever at the point  $A$ ; and let the quantity of matter in the power  $P$  multiplied into the distance of  $A$ , its point of suspension from  $C$ , or  $D$ , be equal to the quantity of matter in the weight  $W$  multiplied into the distance of  $B$  from  $C$ , or  $D$ : it appears from what has been said concerning the weight, that the force with which the power  $P$ , without obstruction, would descend and draw down the point  $A$ , is equal to the force with which the weight  $W$  would descend and draw down the point  $B$ . But, as much force as the power  $P$  exerts to draw down the point  $A$ , it exerts to raise the point  $B$ . Therefore equal and opposite forces are exerted to raise and depress the point  $B$ ; and consequently it will continue at rest, and the weight and power will balance each other.

Case



Case 2. When the power acts obliquely :

Let the weight A hang freely from one end of a balance, so as to have its line of direction DA perpendicular to the arm of the balance; and let another weight as B, be hung at the other end E, in such manner that its line of direction EC, by passing over a pulley at C, may be oblique to the arm of the balance. If the whole force of gravity in the weight B acting in the direction EC, be denoted by the line EC, it may be resolved into two forces denoted by EF and FC, acting in the directions of these lines; of which two forces, the latter only, which acts in the direction FC perpendicular to the arm of the balance, resists the force of gravity in the weight A, the other force FE acting in the direction of the line of the lever. Since therefore, that part of the weight B which acts in opposition to the weight A, is to the whole weight B, as FC to EC; it is manifest, that in order to make the weight B balance the weight A, it must exceed the weight A, in the same ratio that the line EC exceeds the line FC. If from G the center of motion be let fall GH perpendicular to EC, that line will be the perpendicular distance of the direction EC from G; and EG, equal to DG, the perpendicular distance of the direction DA: but the triangles EFC and EHG are similar, because their angles at E are equal, and they have each a right angle; consequently (El. VI. 4.) as EC is to CF, so is EG to HG; but the weight B is to the weight A, as EC to FC; B is therefore to A, as EG, or DG, to HG.

Plate 3.  
Fig. 1.

EXP. I. On the perpendicular surface of a wheel turning on its center of gravity, let weights be suspended at different points; they will be in equilibrio according to the proposition.

2. Let weights and powers be applied to the three kinds of levers, according to the proposition; for example, if the weight be to the power as 3 to 1, let the distance of the point of suspension of the weight be to that of the power as 1 to 3.

3. Apply the same power and weight to a lever of the first kind bended.

4. The common balance is a lever of the first kind, the arms of which, on each side of the prop, are equal. If the center of motion coincides with the center of gravity, that point being supported, the balance rests in any position whatever. If the center of gravity is above the center of motion, the balance, being once put out of its parallel position, can never return again: but if the center of motion is above the center of gravity, equal weights can only sustain each other, when the arms of the balance are in the parallel position, in which case alone their center of gravity is supported.

5. The Roman statera, or steelyard, is a lever of the first kind, whose arms are unequal.

6. The false balance is a lever of the same kind, whose arms are imperceptibly unequal.



DEF. XI. The *Wheel* and *Axis*, is a wheel turning round together with its axis; the power is applied to the circumference of the wheel, and the weight to that of the axis, by means of cords.

## P R O P. LIII.

An equilibrium is produced in the wheel and axis, when the weight is to the power, as the diameter of the wheel to the diameter of the axis.

Plate 3.  
Fig. 2.

Let AB be the diameter of the wheel, DE that of the axis, W the weight, and P the power, suspended from the points D and B. When the wheel has performed one revolution, the power P has drawn off as much cord from the wheel as is equal to its circumference, and has therefore moved through a space equal to that circumference. In the same time the weight W is raised through a space equal to the circumference of the axis, upon which the cord, by which the weight is suspended, is once turned round. Therefore the velocity of the power exceeds the velocity of the weight as much as the circumference, that is the diameter, of the wheel exceeds that of the axis. If then the weight exceeds the power as much as the velocity of the power exceeds that of the weight, that is, as much as the diameter, or semidiameter of the wheel, AB, or CB, exceeds the diameter, or semidiameter of the axis, DE, or CE, the momenta will be equal, and the power and weight will balance each other.

Or thus; The axis and wheel is a lever of the first kind; in which the center of motion is in C, the center of the axis; the weight W, sustained by the rope DW, is applied at the distance DC, the radius of the axis; and the power P, acting in the direction PB perpendicular to CB, the radius of the wheel, is applied at the distance of that radius: therefore, Prop. LII. there is an equilibrium, when the power is to the weight, as the radius of the roller to the radius of the wheel.

EXP. Let the weight and power be as 2 to 1, and the diameter of the wheel to that of the axis in the same proportion.

DEF. XII. The *Pulley* is a small wheel moveable about its axis by means of a cord which passes over it.

P R O P.



## P R O P. LIV.

When the axis of the pulley is fixed, the pulley only changes the direction of the power : if moveable pullies are used, an equilibrium is produced, where the power is to the weight as one to the number of ropes applied to them : if each moveable pulley has its own rope, each pulley will double the power.

If the pulley ED be fixed upon the beam A, the power and weight, in equilibrio, will be equal. But, if one end of the rope be fixed in B, and the other supported by the power P, it is evident, that in order to raise the weight W one foot, the power must rise two; for both the ropes, BC and CP, will be shortened, a foot each; whence the space run over by the power will be double of that of the weight; if therefore the power is to the weight as 1 to 2, their momenta will be equal: for the same reason, if there be four ropes passing from the upper to the lower pullies, the velocity of the power will be quadruple to that of the weight, or as 4 to 1, &c. In all cases, therefore, when the power is to the weight, as 1 to the number of ropes passing from the upper to the lower pullies, there will be an equilibrium.

Plate 3.  
Fig. 3.  
Fig. 4.

Or thus; Every moveable pulley hangs by two ropes equally stretched, which must bear equal parts of the weight; and therefore when one and the same rope goes round several fixed and moveable pullies, since all its parts on each side of the pullies are equally stretched, the whole weight must be divided equally amongst all the ropes by which the moveable pullies hang: consequently, if the power which acts on one rope be equal to the weight divided by that number of ropes, the power must sustain the weight.

Fig. 5.

If each moveable pulley has its own cord, the first, as appears from what has been said, doubles the velocity of the power; and therefore if the power be half of the weight, the momenta will be equal, and the balance will be produced. In like manner, the second pulley causes the weight to move with half the velocity with which it would move, if suspended from the first moveable pulley, that is, makes the velocity of the power quadruple of that of the weight; and so of the rest.

Fig. 6.

Exp. The several kinds of pullies may be shewn to be in equilibrio according to the proposition.

## P R O P. LV.

In the Inclined Plane the power and weight balance each other, when the power is to the weight, as the sine of the inclination of the  
plane



plane is to the sine of the angle which the line of the direction of the power makes with the perpendicular to the plane.

Plate 3.  
Fig. 7.

Let a weight be supported on the inclined plane  $CA$  by a power acting in any given direction  $PD$ . Let the whole force, whereby the weight would descend perpendicularly be represented by  $PB$ : and resolving  $PB$  into two forces, one of which,  $BD$ , is perpendicular to the plane  $CA$ , and the other,  $PD$ , is in the direction of the power; the force  $BD$  is destroyed by the re-action of the plane, and the force  $PD$  will be sustained by an equal power, acting in the direction  $PD$ . Therefore, when there is an equilibrium, the power is to the weight, as  $PD$  to  $PB$ ; that is, as the sine of the angle  $PBD$ , or (El. VI. 8.) its equal  $CAB$ , to the sine of the angle  $PDB$ .

When  $PD$  is in the direction of the plane, this ratio becomes that of  $CD$  to  $CB$ , or of the height of the plane  $CB$ , to  $CA$  its length.

When the direction of the power  $PD$  is parallel to the base of the plane, the ratio of the power to the weight becomes that of  $ED$  to  $EB$ ; or (El. VI. 8. Cor.) of  $CB$ , the height of the plane, to  $BA$  the base.

When the direction of the power coincides with the perpendicular  $BD$ , the ratio of the power to the weight becomes that of the sine of a finite angle, to the sine of an angle indefinitely diminished. From which it appears, that no finite power is sufficient to support a weight upon an inclined plane, if that power acts in a direction perpendicular to the plane.

DEF. XIII. The *Screw* is a cylinder which has either a prominent part, or a hollow line, passing round it in a spiral form, so inserted in one of the opposite kind, that it may be raised or depressed at pleasure, with the weight upon its upper, or suspended beneath its lower, surface.

#### P R O P. LVI.

In the screw the equilibrium will be produced, when the power is to the weight, as the distance between two contiguous threads of the screw to the <sup>spiral line</sup> ~~circumference of the circle~~ described by the power in one revolution of the cylinder.

Plate 3.  
Fig. 8.

While the screw is made to perform one revolution, the weight  $W$  may be considered, as raised up an inclined plane  $cq$ , whose height  $cp$  is the interval between two contiguous spirals, whose base  $pq$  is the periphery of the cylinder, and whose length  $cq$  is the spiral line, by a power acting parallel to the base of the plane: for such an inclined plane, involved about a cylinder, will form the spiral line of the screw. A power at  $p$ , acting parallel



parallel to the base, is in equilibrio with the weight  $W$  to be raised, when the power is to the weight, as the height of the inclined plane, to the base; or, in this case as  $pc$ , the interval between the spirals, to the circumference described by  $p$ : but a power applied at  $P$ , which is to that applied at  $p$ , as the circumference described by  $p$ , to the circumference described by  $P$ , has the same effect; therefore there is an equilibrium, when the power applied at  $P$  is to the weight to be raised, as  $pc$ , the interval between two contiguous spirals, to the circumference described by the power  $P$ .

EXP. Let a weight be raised by a screw according to the proposition.

DEF. XIV. The *Wedge* is composed of two inclined planes, whose bases are joined.

### P R O P. LVII.

When the resisting forces, and the power which acts on the wedge, are in equilibrio, the weight will be to the power, as the height of the wedge to a line, drawn from the middle of the base to one side, and parallel to the direction in which the resisting force acts on that side.

Let the equilateral triangle  $ABC$  represent a wedge, whose base, or back is  $AC$ , whose sides are the lines  $AB$  and  $CB$ , and whose height is the line  $BP$ , which bisects the vertical angle  $ABC$ , and also the base perpendicularly in  $P$ . Let  $E$  and  $F$  represent two bodies, or two resisting forces acting on the sides of the wedge perpendicularly, and whose lines of direction  $EP$  and  $FP$  meet at the middle point of the base, on which the power  $P$  acts perpendicularly, then will  $EP$  and  $FP$  (El. I. 5. and 26.) be equal: let the parallelogram  $ENFP$  be completed; its diagonals  $PN$  and  $EF$  will bisect each other perpendicularly in  $H$ . Now when these forces (which act perpendicularly on the sides and base of the wedge) are in equilibrio, they will be to each other (Prop. XIV.) as the sides and diagonal of this parallelogram, that is, the sum of the resisting forces will be to the power  $P$ , as the sides  $EP$  and  $FP$  to the diagonal  $PN$ , or as one side  $EP$  to half the diagonal  $PH$ , that is (from the similarity of the right-angled triangles  $BEP$ ,  $EHP$ ) as  $BP$ , the height of the wedge, to  $EP$  the line which is drawn from the middle of the base to the side  $AB$ , and is the direction in which the resisting force acts on that side.

From the demonstration of this case, in which the resisting forces act perpendicularly on the sides of the wedge, it appears that the resistance is to the power which sustains it, as one side of the wedge  $AB$  is to the half of its breadth  $AP$ ; because  $AB$  is to  $AP$ , (El. VI. 8.) as  $BP$  is to  $EP$ .

It

Plate 3.  
Fig. 9.



It appears also from hence, that if  $PN$  be made to denote the force with which the power  $P$  acts on the wedge, the lines  $PE$  and  $PF$  which are perpendicular to the sides, will denote the force with which the power  $P$  protrudes the resisting bodies in directions perpendicular to the sides of the wedge.

Let us now suppose, in the second case, that the resisting bodies  $E$  and  $F$  act upon the wedge in directions parallel to the lines  $DP$  and  $OP$ , which are equally inclined to its sides, and meet in the point  $P$ . Draw the lines  $EG$  and  $FK$  perpendicular to  $DP$  and  $OP$ ; then making  $PN$  denote the force with which the power  $P$  acts on the wedge,  $PE$  and  $PF$  will denote the forces with which it protrudes the resisting bodies in directions perpendicular to the sides of the wedge, as was observed before; now each of these forces may be resolved into two, denoted respectively by the lines  $PG$  and  $GE$ ,  $PK$  and  $KF$ , of which  $GE$  and  $KF$  will be lost, as they act in directions perpendicular to those of the resisting bodies; and  $PG$  and  $PK$  will denote the forces by which the power  $P$  opposes the resisting bodies, by protruding them in directions contrary to those in which they act on the wedge; therefore when the resisting forces are in equilibrio with the power  $P$ , the former must be to the latter, as the sum of the lines  $PG$  and  $PK$  is to  $PN$ , or as  $PG$  is to  $PH$ . But (El. VI. 4.)  $PG$  is to  $PE$ , as  $PE$  to  $PD$ ; and  $PH$  is to  $PE$ , as  $PE$  to  $PB$ : whence (El. VI. 16.) both the rectangles  $PG, PD$ , and the rectangles  $PH, PB$ , are equal to the square of  $PE$ ; these rectangles are therefore equal to one another: whence their sides (El. VI. 14.) are reciprocally proportional, that is,  $PG$  is to  $PH$ , as  $PB$  to  $PD$ . Whence it follows from what was shewn above, that, in equilibrio, the resisting forces are to the power, as  $PB$  to  $PD$ ; that is, as the height of the wedge to the line drawn from the middle of the base to one side of the wedge, and parallel to the direction in which the resisting force acts on that side.

From what has been demonstrated, we may deduce the proportion of the power to the resistance it is able to sustain in all the cases in which the wedge is applied. First, when in cleaving timber the wedge fills the cleft, then the resistance of the timber acts perpendicularly on the sides of the wedge, therefore in this case, when the power which drives the wedge, is to the cohesive force of the timber, as half the base, to one side of the wedge, the power and resistance will be in equilibrio.

Secondly; When the wedge does not exactly fill the cleft, which generally happens, because the wood splits to some distance before the wedge. Let  $ELF$  represent a cleft into which the wedge  $ABC$  is partly driven; as the resisting force of the timber must act on the wedge in directions perpendicular to the sides of the cleft, draw the line  $PD$  in a direction perpendicular to  $EL$  the side of the cleft, and meeting the side of the wedge in  $D$ ; then the power driving the wedge and the resistance of the timber, when they balance, will be to each other as the line  $PD$  to  $PB$  the height of the wedge.

Thirdly; When a wedge is employed to separate two bodies that lie together on an horizontal plane, for instance two blocks of stone; as these bodies must recede from each other in horizontal directions, their resistance must act on the wedge in lines parallel to its base  $CA$ ; therefore the power which drives the wedge, will balance the resistance

Exp.



when they are to each other as PA, half the breadth of the wedge, to PB its height; and then any additional force, sufficient to overcome the resistance arising from the friction of the bodies on the horizontal plane, will separate them from each other.

EXP. Let a power and weight be applied to a wedge according to the proposition.

SCHOL. 1. Since in all the mechanical powers, an equilibrium is produced when the power is to the weight as the velocity of the weight is to the velocity of the power, in all compound machines there will be an equilibrium, when the sum of the powers are to the weight, as the velocity of the weight is to the sum of the velocities of the powers.

EXP. 1. Let A, B, C, be a compound lever, consisting of three levers, in the first of which, A, the velocity of the weight is to that of the power, as 1 to 5; in the second, B, as 1 to 4; in the third, C, as 1 to 6. The velocity of the weight will be to that of the power, as 1 to  $5 \times 4 \times 6 = 120$ ; and if the power be to the weight, as 1 to 120, they will balance each other. *Plate 2. Fig. 15.*

2. Let GC and LF be levers fixed to the supporters RA, SE, and let their shorter arms be kept in equilibrio with the longer respectively by the weights fixed at G and L. Let NH be a bar screwed to the fixed parts to keep them steady. If the power C be ten times farther from A the prop, than the weight B, they will be in equilibrio when the power C is to the weight P, as 1 to 10. In like manner, the distance ME being ten times DE, if the power M be  $\frac{1}{10}$  of the weight C suspended from D, they will be in equilibrio: whence M, 1, will balance P, 100. *Fig. 16.*

3. Exhibit models or draughts of different compound machines, as mills, cranes, the pile-driver, &c.

SCHOL. 2. The inequality of the surface on which any body moves occasions an attrition, called friction, which prevents the accurate agreement of many experiments in mechanics with theory. This resistance is always in the direct ratio of the roughness of the surface, and the weight of the body.

EXP. 1. The same weight will require the same force to carry it over the same obstacle, whether the touching surface be greater or less: this will appear by drawing a piece of smooth wood, having a broad and a narrow surface, upon each surface successively, by means of a cord, weight and pulley.

2. Let the same body be drawn over surfaces of different degrees of roughness.

SCHOL. 3. Wheel carriages are used, to avoid friction as much as possible. A wheel turns round upon its axis, because the several points of its circumference are retarded in succession by attrition, whilst the opposite points move freely. Large wheels meet with less resistance than smaller from external obstacles, and from the friction of the axle, and are more easily drawn, having their axles level with the horses. But in uneven roads, small wheels are used, that in ascents the action of the horse may be nearly parallel with the plane of ascent, and therefore may have the greatest effect: small wheels are also



more conveniently turned. The greater part of the load should be laid on the hinder part of a wheel carriage.

- EXP. 1. Draw 4 large wheels along a plain surface; then 2 large and 2 small wheels.  
 2. Draw them successively over obstacles.  
 3. Draw a carriage with two-thirds of the load forward; then with the same on the hinder part.

## C H A P. VII.

### *Of Motion as produced by the united Forces of PROJECTION and GRAVITATION.*

#### S E C T. I.

#### *Of Projectiles.*

#### P R O P. LVIII.

Bodies thrown horizontally or obliquely, have a curvilinear motion, and the path which they describe is a parabola.

Plate 3.  
Fig. 10,  
11.

If a body be thrown in the direction AF, and acted upon by the projectile force alone, it will continue to move on uniformly in the right line AF, and would describe equal parts of the line AF in equal times, as AC, CD, DE, &c. But if, in any indefinitely small portion of time, in which the body would by the projectile force move from A to C, it would by the force of gravity have fallen from A to G; by the composition of these forces (Prop. XVI.) it will, at the end of that time, be found in H, the opposite angle of the parallelogram ACGH. In two such portions of time, whilst it would have moved from A to D by the projectile force, it would (Prop. XXVI.) by gravitation fall through four times AG, that is, AM; and therefore, these forces being combined, it will be found at the end of that time in I, the opposite angle of the parallelogram DM. In like manner, at the end of the third portion of time, it would by the projectile force be carried through three equal divisions to E, and by the force of gravitation over nine times AG to N; and consequently, by both these forces acting jointly, it will be carried to K,  
 the



the opposite angle of the parallelogram EN. Therefore the lines CH, DI, EK, that is, AG, AM, AN, which are to each other as the numbers 1, 4, 9, are as the squares of the lines AC, AD, AE, that is, GH, MI, NK, which are as 1, 2, 3. And because the action of gravitation is continual, the body in passing from A to H &c. is perpetually drawn out of the right line in which it would move if the force of gravitation were suspended, and therefore moves in a curve. And H, I and K are any points in this curve in which lines let fall from points equally distant from A in the line AB meet the curve. Therefore the body moves in a parabola, the property of which is, (*Simson's Conic Sections*, Book I. Prop. XII. Cor.) that the *abscissa* AG, AM, AN are to each other as the squares of the ordinates GH, MI, NK.

EXP. I. Let a small ball of ivory be let fall along a smooth surface, of polished brass or tin, of a concave circular form, fastened upon a board placed perpendicularly; the ball, acquiring an horizontal projection, will fall in the curve of a parabola, as may be seen by placing a board, having this curve drawn upon it, near the path of the ball.

2. The same may be shewn with water spouting from a pipe; or by a ball revolving on a whirling table, and suddenly set at liberty by cutting its cord.

## P R O P. LIX.

The path which a body thrown perpendicularly upwards describes in rising and falling is a parabola.

A stone lying upon the surface of the earth, partaking of the motion of the earth (here supposed) round its axis, this motion which it has with the earth will not be destroyed by throwing it in a direction perpendicular to the surface of the earth. After the projection, therefore, the stone will be moved by two forces, one horizontal, the other perpendicular, and will rise in a direction which may be shewn as in the last proposition to be the parabolic curve; in which it will continue till it reaches the highest point, from whence it might be shewn, as in the last proposition, that it will descend through the other side of the parabola.

## P R O P. LX.

The velocity with which a body ought to be projected to make it describe a given parabola, is such as it would acquire by falling through a space equal to the fourth part of the parameter belonging to that point of the parabola from which it is intended to be projected.

The velocity of the projectile at the point A (by Prop. LVIII.) is such as would carry it from A to E, in the same time in which it would descend by its gravity from A to N.

I 2

and

Plate 3.  
Fig. 10,  
11.



And the velocity acquired in falling from A to N (by Prop. XXVII.) is such as in the same time by an uniform motion would carry the body through a space double of AN. Therefore the velocity which is acquired by the body in falling to N is to that with which the body is projected at A and uniformly carried forwards to E, as twice AN is to AE. But since, from the nature of the parabola, (*Simson's Conic Sections*, Book I. Prop. XIII.) the square of AE divided by AN ( $\frac{AE^2}{AN}$ ) is equal to the parameter of the point A, one fourth part of this parameter will be expressed by  $\frac{\frac{1}{4}AE^2}{AN}$ . And because the velocities acquired by falling bodies are (by Prop. XXVI. Cor. 1.) as the square roots of the spaces they fall through, the velocity acquired by a body in falling through AN is to the velocity acquired in falling through  $\frac{\frac{1}{4}AE^2}{AN}$  or one fourth part of the parameter of A, as the square root of AN to the square root of  $\frac{\frac{1}{4}AE^2}{AN}$ ; that is, as  $\sqrt{AN}$  to  $\frac{\frac{1}{2}AE}{\sqrt{AN}}$ , or (multiplying each term by  $\sqrt{AN}$ ) AN to  $\frac{1}{2}AE$ , or twice AN to AE. Therefore the velocity acquired by a body in falling from A to N has the same ratio to the velocity with which the body is projected or the line AE described, and to the velocity acquired by a body in falling through a fourth part of the parameter belonging to the point A: consequently (El. V. 11.) these velocities are equal.

Plate 3.  
Fig. 13.

COR. Hence may be determined the direction in which a projectile from a given point, with a given velocity, must be thrown to strike an object in a given situation.

Let A be the place from which the body is to be thrown, and K the situation of the object. Raise AB perpendicular to the plane of the horizon, and equal to four times the height from which a body must fall to acquire the given velocity. Bisection AB in G: through G draw HG perpendicular to AB: at the point A raise AC perpendicular to AK, and meeting HG in C: on C as a center with the radius CA describe the circle ABD; and through K draw the right line KEI perpendicular to the plane of the horizon, and cutting the circle ABD in the points E and I. AE, or AI, will be the direction required.

For, drawing BI, BE, since AK is a tangent to the circle, and BA, IK, are parallel to each other, the angle ABE (El. III. 32.) is equal to the angle EAK; and the alternate angles BAE, AEK, are equal: therefore the triangles ABE, AEK are similar; and AB is to AE, as AE to EK. Therefore  $AB \times EK = AE^2$ ; and (dividing both sides of the equation by EK)  $AB = \frac{AE^2}{EK}$ . In like manner, the triangles BAI, KAI,

being similar, BA is equal to  $\frac{AI^2}{IK}$ . Since then, AB is equal to four times the height from which a body must fall to acquire the velocity with which it is to be thrown;  $\frac{AE^2}{EK}$  (or  $\frac{AI^2}{IK}$  its equal) is the same. Consequently (by this Prop.) the point K will be in the parabola which the body will describe, which is thrown with the given velocity in the directions AE, or AI, and the body will strike an object placed at K.

SCHOL.



SCHOL. If the velocity with which a projectile is thrown be required, it may be determined from experiments in the following manner. By the help of a pendulum or any other exact chronometer, let the time of the perpendicular flight be taken; then, since the times of the ascent and descent are equal, the time of the descent must be equal to one half of the time of the flight, consequently, that time will be known: and, since a heavy body descends from a state of rest at the rate of 16 feet in the first second of time, and that the spaces through which bodies descend are as the squares of the times; if we say, as one second is to 16 feet, so is the square of the number of seconds which expresses the time of the descent of the projectile, to a fourth proportional, we shall have the number of feet through which the projectile fell, which being doubled, will give us the number of feet which the projectile would describe in the same time with that of the fall, supposing it moved with an uniform velocity, equal to that which it acquired by the end of the fall; which last found number of feet, being divided by the number of seconds which expresses the time of the projectile's descent, will give a quotient, expressing the number of feet, through which the projectile would move in one second of time with a velocity equal to that which it acquired in its descent, which velocity is equal to the velocity with which the projectile was thrown up; consequently, this velocity is discovered.

## P R O P. LXI.

The squares of the velocities of a projectile in different points of its parabola, are as the parameters belonging to those points.

For (by the last Prop.) the velocities in the several points of the parabola, are equal to the velocities acquired in falling through the fourth parts of the parameters of the points. Therefore the squares of these velocities being (by Prop. XXVI.) as the spaces described, the squares of the velocities in the several points of the parabola are as the fourth parts of the parameters of those points: but the whole parameters are as their fourth parts: therefore the squares of the velocities at the several points of the parabola are as the parameters of those points.

COR. Hence, setting aside any difference which may arise from the resistance of the air, a projectile will strike a mark as forcibly at the end as at the beginning of its course, if the two points be equally distant from the principal vertex: for, the parameters belonging to these points being equal, the velocities in these points must also be equal.

## P R O P. LXII.

When a body is thrown obliquely with a given velocity, if the space through which it must have fallen perpendicularly to acquire  
that



that velocity is made the diameter of a circle, the height to which the body will rise is equal to the versed sine of double the angle of elevation.

Plate 3.  
Fig. 12.

Let a body be thrown in the direction BE, with the same velocity which any body would acquire by falling perpendicularly through AB; if AB is made the diameter of a circle, the greatest height to which the body will rise will be BD.

Let IL be a right line drawn in the plane of the horizon touching the circle in B, and making with the line BE, which is the direction in which the body is thrown, the angle IBE, or angle of elevation. Because IL touches the circle, and EB drawn in the circle meets it in the point of contact (El. III. 32.) the angle EBI is equal to the angle EAB. And ECB is double of EAB (El. III. 20.) therefore ECB is double of EBI, the angle of elevation. And BD is the versed sine of ECB, that is, of double the angle of elevation.

Let BE represent the velocity with which the body is thrown. Then since this velocity is, by supposition, such as might be acquired by falling down AB, if the body was thrown perpendicularly upwards with the same velocity BE, it would rise to the height BA. Let the oblique motion BE be resolved into two others, one in the direction BD perpendicular to the horizon, and the other in the direction DE parallel to it: then the ascending velocity will be to the horizontal velocity, as BD to DE, and to the whole velocity, as BD to BE. But the part of the velocity BD is the only part which is employed in raising the body, since the other part DE is parallel to the plane of the horizon. Now, the height of a body ascending perpendicularly with the whole velocity BE, will be to the height when it ascends with the part BD (compare Prop. XXVI. and Prop. XXVIII.) as the square of BE to the square of BD. But because (El. VI. 8.) the triangle EDB is similar to the triangle AEB, BD is to EB, as EB is to BA; and BD, BE, BA, being continued proportionals, BD is to BA, as the square of BD is to the square of BE. And the perpendicular heights to which the velocities BE and BD will make the body ascend have been shewn to be as the square of BE to the square of BD; the heights are therefore as BA to BD. Since therefore the first velocity BE would make the body ascend through BA, the other velocity BD, which is the part of the whole velocity which acts to make the body thrown in the direction DE to ascend, will carry it to the height BD, which is the versed sine of double the angle of elevation. The same might be shewn in any other direction of the body, as BF, or BG.

DEF.



DEF. XV. The *Random* of a projectile is the horizontal distance to which a heavy body is thrown.

## P R O P. LXIII.

When a body is thrown obliquely with a given velocity, if the space through which it must have fallen perpendicularly to acquire that velocity is made the diameter of a circle, the random will be equal to four times the sine of double the angle of elevation.

If EBI be the angle of elevation, and ECB double that angle, DE will be the sine of double the angle of elevation. Let a body be thrown from the point B in the direction BE, with the velocity which it would acquire in falling through AB; the random, or horizontal distance at which the body will fall, is equal to four times DE.

Plate 3.  
Fig. 12.

For, since (as in the last Prop.) the velocity BE being resolved in BD, DE, the ascending velocity is BD, and the horizontal DE, if these two velocities were to continue uniform, the spaces described in equal times (Prop. V.) would be as the velocities, and in the same time in which the body by the ascending velocity would rise through BD, by the horizontal velocity it would be carried forwards through DE. Of these velocities, the horizontal one DE is uniform, because the force of gravity can neither accelerate nor retard a motion in this direction; but the ascending velocity is uniformly retarded; and therefore the body (compare Prop. XXIII. and XXVIII.) will be twice as long in ascending to its greatest height BD, as it would have been if the first ascending velocity had continued uniform: but on this supposition, the body would have been carried through BD and DE in the same time: therefore in double the time, that is, in the time of ascent through BD with an uniformly retarded velocity, it would be carried forward through twice DE: consequently, in the times of descent and ascent together it would move forwards through four times DE. Therefore a body thrown from B in the direction BE with such a velocity as might be acquired by falling down AB, the diameter of a circle, will fall at the distance of four times the sine of double the angle of elevation.

## P R O P. LXIV.

The random of a projectile will be the greatest possible, with a given velocity, when the angle of elevation is an angle of forty-five degrees.

The velocity being given, the height from whence the body must have fallen to acquire that velocity, or (Prop. XXXV.) the diameter of the circle AB, is a given quantity.

And



And in a given circle the greatest sine is the radius or sine of a right angle : therefore four times the radius is greater than four times any other sine ; and consequently, the random which is equal to four times the radius, (which, by Prop. LXII. will be the case when the double angle of elevation is a right one, or the angle of elevation forty-five degrees) will be the greatest possible random.

Exp. This proposition may be illustrated by water spouting from a pipe.

### P R O P. LXV.

The random of a projectile, whose velocity is given, will be the same at two different elevations, if the one be as much above forty-five degrees as the other is below it.

Plate 3.  
Fig. 12.

If EBI be an angle of 30 degrees, and GBI an angle of 60 degrees, because EBI falls short of half a right angle as much as GBI exceeds it, the double of EBI will fall short of a right angle as much as the double of GBI will exceed it, therefore, from the definition of a sine, these doubles will have the same sine. Consequently, four times their sines, that is (by Prop. LXIII) their randoms will be equal.

Exp. This proposition may be illustrated as the preceding.

### P R O P. LXVI.

The greatest random of a projectile, whose velocity is given, is double the height to which it would rise if it were thrown perpendicularly with the same velocity.

Plate 3.  
Fig. 12.

If a body be projected in the direction BF, at an angle of forty-five degrees, and its velocity be equal to that which a body would acquire in falling down AB (by Prop. LXIV.) the random will be the greatest possible, and will be equal to four times CF, or twice BA. But the body cast perpendicularly upwards with the same velocity would (by Prop. XXVIII.) rise to the height BA. Therefore the greatest random, with a given velocity, is double the height to which the body, thrown perpendicularly with the same velocity, would rise.

Exp. By spouting fluids as before.

### P R O P. LXVII.

The randoms of projectiles, whose elevations are given, are as the squares of their velocities.

If



If a body be thrown in any direction BE, its random (Prop. LXIII.) will be equal to four times DE, or four times the sine of double the angle of elevation, in a circle whose diameter AB is the height from which the body must fall to acquire the velocity with which it is projected. But the velocities being supposed variable, AB the diameter will be directly as the velocity; since the greater velocity a body moves with, the greater space it will fall through in a given time. And because, in the triangle EDC the angle at D being a right angle is always invariable, and that the angle ECD, which is double of EAD, that is (El. III. 32.) of the given angle of elevation EBI, is given, the triangle ECD in every variation of AB, is always equiangular and similar to itself, and ED is always as EC: but EC being a radius, is as AB: therefore ED, the sine of the given angle of elevation, is as AB the diameter. Consequently four times the sine ED, that is, the random is as AB. But the height AB from which a body must fall to acquire any velocity, is (by Prop. XXVI.) as the square of that velocity. Therefore the random is as the square of the velocity.

Plate 3.  
Fig. 12.

## S E C T. II.

### *Of Central Forces.*

#### P R O P. LXVIII.

A body which is constantly drawn or impelled towards any point, may be made to describe, round that point as a center, a curve returning into itself.

Let T be the center of the earth, and GDEI its surface. Let a body be projected in any direction, GH, which does not pass within the surface of the earth. The projectile force together with the force of gravity will make it describe a curve, which, as the projectile force is increased, will recede farther from the perpendicular GA, as GB, GC, GD. It is manifest that the projectile force may be increased, till the body shall pass beyond the surface CDKE, and move in the path GML, GNV, or some larger curve.

Plate 3.  
Fig. 13.

First; Suppose the projectile force to be such, that the body will be carried in the semicircle GN, it will continue in the curve of that circle till it returns to G. For, when a body moves in the circumference of a circle, (as in fig. 15.) the projectile force,

K

acting



Plate 3.  
Fig. 15.

acting in a line which is a tangent to the circle, as GB, acts (El. III. 18.) in a direction which is perpendicular to the direction BA, in which it is impelled towards the center. And since, if the force which impels the body towards the center ceased to act in any point, as C, the body would move forwards in the right line CF, the projectile force in every point of the circumference, acts in a direction perpendicular to the force of gravitation: consequently, these two forces remaining the same, and acting always in the same direction with respect to each other, the velocity of the body must remain the same: whence, at the point M, it will have the same power to recede from the center as at G; and, retaining this power through every remaining part of its course, it will proceed in the circumference, till it arrive at G, and will continue to revolve in the circle.

Fig. 14.

Next; let the body be projected from G with a force greater than that which is required to carry it round in the circumference of the circle GNV; and let the curve in which it moves be an ellipse, having the earth in its remoter focus. Because the force of projection, as the body proceeds in the first half of its orbit, acts in the direction of a tangent to the curve, whilst the force of gravitation acts in the direction of a right line from the body to the center of the earth, the directions of these two forces make an acute angle with one another, and consequently, through this part of the course of the body, the force of gravitation *conspiring with* the force of projection, the velocity of the body must be increased, and at the same time it must be continually *drawn downwards* towards the earth. At the point in which the forces act in directions perpendicular to each other, the force of gravitation does not conspire with that of projection to bring the body towards the earth: and afterwards, in the latter half of its course, the directions of the forces making an obtuse angle with each other, the force of gravitation is *opposed* by that of projection in the same degree in which the former was before aided by the latter; and therefore the body in passing towards G will fly off from the earth or *rise*, as much as it before approached to the earth or *descended*, and thus will return to the point G with the same velocity with which it set out at first, having lost as much velocity by receding from the earth in the latter part of its course, as it had gained by falling towards the earth in the former part.

Lastly; Let the body be projected from G with a force which is less than sufficient to carry it round in the circle GNV; and let it perform its revolutions in an elliptic curve whose greater axis is less than the radius of the circle GNV, setting out from G, and having the earth in the nearer focus: the effect will be the same as in the last case, except that the projectile force will oppose the force of gravitation in the *first* half of the revolution, and conspire with it in the *latter*.

Exp. Let a ball revolve round the central point of a Whirling Table. Concerning the construction and use of this machine, see Ferguson's Lectures, Lect. II.



## P R O P. LXIX.

A body revolving in an orbit, endeavours in every point of its course to fly off from the center in a right line which is a tangent to the orbit.

Let BCDL be a circle in which a body is revolving: when it is arrived at the point B, let the force which impels it towards the center be withdrawn, and the body (by Prop. I.) would fly off from the point B in the direction BG: in like manner at C, it would fly off in the right line CF; at D, in DH; and at L, in LH. The same is manifestly true in an elliptical orbit. Now the same force with which it would fly off, if no other cause prevented it, must make it endeavour to fly off in the same manner in every point of the orbit.

Plate 3.  
Fig. 15.

EXP. Whilst a ball is revolving on a whirling table, if the cord which retains it, be suddenly cut, the ball will fly off in a right line, which will be a tangent to the orbit in which it moved.

COR. A body revolving about a center endeavours to recede from that center: for every point of the tangent in which it endeavours to move out of the circle, is farther from the center, than the point in which the tangent meets the curve.

EXP. I. Let a ball, laid on the whirling table and connected with a weight which hangs freely by a cord passing through the center of the table, be put into motion; it will fly off from the center.

2. An open vessel, containing water, may be revolved in a plane perpendicular to the horizon without losing any of the water.

DEF. XVI. The force which impels a body towards the center, when it revolves in an orbit, is called the *centripetal* force; that by which it endeavours to recede from the center, is called the *centrifugal* force; and these two forces are called jointly, *central* forces.

SCHOL. The projectile and centrifugal forces differ from each other, as the whole from the part. The projectile force is that with which a body would move forwards in a tangent to its orbit, if there were no centripetal force to prevent it: the centrifugal force is that part of the projectile force which carries the body off from the center while it is describing the tangent. Thus, if the body revolved in the orbit BD, the projectile force is that which would make it describe the tangent BA, if the centripetal force were to cease acting. But in the mean time, the whole force BA does not carry the body off from the

Plate 3.  
Fig. 16.



center C: when it is arrived at A, it is farther from the center than it was at B, only by the length AN, and it is that part of the projectile force which, when the whole is resolved into two forces, may be considered as acting in this line AN, which carries the body off from the center, and is called the centrifugal force.

## P R O P. LXX.

*circular*

When bodies revolve in an orbit about a center, the centripetal and centrifugal forces are equal.

Plate 3.  
Fig. 16.

If a body revolve in the circle BD, in the time in which it describes the arc BN, it will have been impelled towards the center through the space AN; for, by the projectile force alone it would have been carried from B to A. The line AN is then the space described by means of the centripetal force, and this force is proportional to AN. But if, when the body was at B, no centripetal force had acted upon it, instead of describing the arc BN, it would have moved along the tangent BA, and the line NA would have been the space through which it would have departed from the center: therefore the centrifugal force is proportional to NA. Both these forces being then proportional to the same line NA, they are equal to one another.

## L E M M A I.

*Quantities, and the ratios of quantities, which, in any finite time, tend continually to equality, and, before the end of that time, approach nearer to each other than by any given difference, become ultimately equal.*

If you deny it, let them be ultimately unequal; and let their ultimate difference be D. Therefore they cannot approach nearer to equality than by that given difference D; which is contrary to the supposition. If a straight and a curve line, continually diminishing, perpetually approach towards equality, and at the end of any finite time would vanish together, at the instant in which they are vanishing they are equal.

## L E M. II.

Plate 3.  
Fig. 18.

*If in any figure AacE, terminated by the right lines Aa, AE, and the curve acE, there are inscribed any number of parallelograms Ab, Be, Cd,*



*Cd, &c. contained under equal bases AB, BC, CD, &c. and the sides, Bb, Cc, Dd, &c. parallel to Aa the side of the figure; and the parallelograms aKbl, bLcm, cMdn, &c. are completed: then, if the breadth of these parallelograms be diminished, and their number augmented continually, the ultimate ratios, which the inscribed figure AKbLcMdD, the circumscribed figure AalbmcndoE, and the curvilinear figure AabcdE, have to each other, are ratios of equality.*

For the difference of the inscribed and circumscribed figure is the sum of the parallelograms Kl, Lm, Mn, Do, that is (because of the equality of all their bases) the rectangle under one of their bases Kb, and the sum of their altitudes Aa; that is, the rectangle ABla. But this rectangle, because its breadth AB is diminished indefinitely, becomes less than any given rectangle. Therefore (by Lem. I.) the inscribed and circumscribed, and much more the intermediate curvilinear figure become ultimately equal.

## L E M. III.

*The same ultimate ratios are also ratios of equality, when the breadths AB, BC, CD, &c. of the parallelograms are unequal, and are all diminished indefinitely.*

Plate 3.  
Fig. 18.

For, let AF be equal to the greatest breadth; and let the parallelogram FAaf be completed. This will be greater than the difference of the inscribed and circumscribed figures; but, because its breadth AF is diminished indefinitely, it will become less than any given rectangle.

COR. Hence the ultimate sum of the evanescent parallelograms coincides in every part with the curvilinear figure. Much more does the rectilinear figure, which is comprehended under the chords of the evanescent arcs *ab, bc, cd, &c.* ultimately coincide with the curvilinear figure. As also the circumscribed rectilinear figure, which is comprehended under the tangents of the same arcs. And therefore, these ultimate figures (as to their perimeter *acE*) are not rectilinear, but curvilinear limits of rectilinear figures.

## P R O P. LXXI.

The plane of an orbit in which a body revolves passes through the line of projection, and through the center towards which the centripetal force is directed.

Let



Plate 4.  
Fig. 2.

Let  $ABCF$  be the orbit in which the body revolves;  $S$  the center, or point towards which the centripetal force is directed; and  $AV$  the line of projection: the plane of the orbit will pass through  $AV$  and  $S$ ; or the orbit lies in the same plane, with the lines  $AV$ ,  $Bd$ , &c. (lines in which the projectile force acts in different parts of the orbit) and with the center  $S$ .

For, let  $ABCD$  represent a part of the orbit described by a body impelled towards  $S$ . The body beginning to move by the projectile force from  $A$  in the direction  $ABV$ , would, by that force alone, be carried on uniformly in that direction. Suppose the centripetal force to act upon it by separate impulses after equal intervals of time, and that, when the body is carried by the projectile force to  $B$ , it receives one impulse from the centripetal force, drawing it out of its course towards  $S$ , so that by the action of both forces together at  $B$ , it will (by Prop. XIV.) be made to describe  $BC$  in the same time in which the projectile force alone would have made it describe  $Bc$ . The same will take place after equal intervals at  $C$  and  $D$ . At  $B$ , the projectile force is in the direction  $BV$ , the centripetal force in the direction  $BS$ . Let  $Bc$  and  $BG$ , taken in the directions of these forces, represent their ratio to each other, and the projectile force be to the centripetal as  $Bc$  to  $BG$ . The body (by Prop. XIV.) will describe  $BC$ , the diagonal of a parallelogram of which  $Bc$  and  $BG$  are the sides. But (El. XI. 1 and 2.)  $BC$  is in the same plane with  $Bc$  and  $BG$ , that is, with  $AV$ , the line of projection, and with  $BS$ , in which is the center or point  $S$ . The same may be proved concerning the lines  $CD$ , and  $DE$ . And if the projectile force acts continually and not by interrupted impulses, the diagonals  $AB$ ,  $BC$ , &c. will be diminished indefinitely, and the ultimate perimeter  $ADE$  (by Lem. III. Cor.) will become a curve line, which, from what hath been shewn, must be always in the same plane with the line of projection and with the center.

### P R O P. LXXII.

A body revolving in an orbit describes, by a radius drawn to the point towards which the centripetal force acts, equal areas in equal times, and in unequal times areas proportional to the times.

Plate 4.  
Fig. 2.

Let  $ABCD$ . be part of an orbit described by a body which revolves round the point  $S$ , towards which it is impelled by a centripetal force. If this force be supposed to act upon the body by separate impulses, as at  $B$ ,  $C$ ,  $D$ , when the body receives the impulse at  $B$ , it will be drawn out of its course towards  $S$ , and (by Prop. XIV.) will describe the diagonal  $BC$  in the same time in which the projectile force alone would have made it describe  $Bc$ . After equal intervals, the same will take place at  $C$ , and at  $D$ .

Since  $AB$ ,  $BC$ , &c. are the lines described in equal times by the body, the areas described round  $S$  by a radius drawn from the body to  $S$ , are  $ASB$ ,  $BSC$ , &c. Now  $AB$ ,  $Bc$ , expressing spaces passed over in equal times by the uniform motion of the body acted upon

by



by the projectile force alone, are equal bases of the triangles  $ASB$ ,  $BS_c$ , which, being terminated by the same point  $S$ , are of the same altitude: these triangles are therefore (El. I. 38.) equal. And, because the body at  $B$ , is by the joint action of the projectile and centripetal forces carried forwards in the diagonal  $BC$ , of the parallelogram  $Gc$ , the opposite sides thereof,  $GB$ ,  $Cc$ , are parallel; and  $Cc$  is parallel to  $BS$ . But  $BS$  is the common base of the two triangles  $BSC$ ,  $BS_c$ . Therefore these triangles, being upon the same base and between the same parallels, (El. I. 37.) are equal. Consequently,  $ASB$ , which has been proved equal to  $BS_c$ , is likewise equal to  $BSC$ ; that is, the areas described in equal times are equal. And, by composition, any sums of these areas  $ASC$ ,  $ASE$ , are to each other as the times in which they are described; that is, universally, the areas are as the times.

Let the number of these triangles be augmented, and their breadth diminished indefinitely, and (by Lem. III. Cor.) their ultimate perimeter will be a curve line; and therefore the centripetal force will act continually; and, the above reasoning still being applicable to those triangles whose breadth is indefinitely diminished, the areas will be as the times.

COR. 1. The velocity of a body revolving freely about an immoveable center is inversely as a perpendicular let fall from that center on a right line that touches the orbit. For since the lines  $AB$ ,  $BC$  are described in equal times, the velocities will be as these lines, which being (by this Prop.) the bases of equal triangles, must be inversely as the heights of the triangles; therefore the velocities are inversely as these heights, which are measured by perpendiculars let fall from the common vertex, the center  $S$ , to the bases, or the bases produced, that is, when  $AB$ ,  $BC$ , &c. are indefinitely small, to a tangent to the orbit.

COR. 2. If the chords  $AB$ ,  $BC$ , of two arcs successively described in equal times by the same body moving freely, are completed into a parallelogram  $ABCG$ , and the diagonal  $BG$ , in the position which it ultimately acquires when these arcs are diminished indefinitely, be produced towards  $S$ , it will pass through  $S$ , the center of the centripetal force; for  $BG$  is an indefinitely small part of the radius  $SB$ .

COR. 3.  $AB$ ,  $BC$ , and  $DE$ ,  $EF$ , of arcs described in equal times be completed into the parallelograms  $ABCG$ ,  $DEFZ$ , the centripetal forces at  $B$  and  $E$  will be to each other in the ultimate ratio of the diagonals  $BG$ ,  $EZ$ , when those arcs are indefinitely diminished. For, the motions of the body,  $BC$ ,  $EF$ , (by Prop. XVI.) are compounded of the motions  $Bc$ ,  $BG$ , and  $Ef$ ,  $EZ$ ; but  $BG$  and  $EZ$  are equal to  $Cc$  and  $Ff$ , which, as appears from this proposition, are generated by the impulses of the centripetal force in  $B$  and  $E$ , and are therefore proportional to those impulses.

COR. 4. The forces with which bodies are drawn into curvilinear orbits are to each other as the versed sines,  $\frac{1}{2}GB$ ,  $\frac{1}{2}ZE$ , of the indefinitely small arcs  $AC$ ,  $DF$ , described in equal times, which versed sines converge to the center  $S$ , and bisect the chords when those arcs are diminished indefinitely: for such versed sines are half the diagonals of a parallelogram,  $BG$ ,  $EZ$ , being bisected by the diagonals  $AC$ ,  $FD$ .

PROP.



## P R O P. LXXIII.

When a body describes equal areas in equal times about an immovable point, or proportional areas in unequal times, it is impelled to wards that point by the centripetal force which retains it in its orbit.

Plate 4.  
Fig. 2.

Let AB, BC, &c. be lines described by the revolving body in equal times; and it may be proved as before, that the triangles ASB, and BSc, being of the same height, and having equal bases AB, Bc, are equal. But, by supposition, ASB, BSC, are equal; therefore BSc and BSC are equal. And these equal triangles, being upon the same base SB, (El. I. 39.) are between the same parallels; therefore Cc and BS are parallel. Because the body at B is acted upon by two forces, the projectile force in the line Bc, and the centripetal force, and, by supposition, these two forces together make it describe BC; BC is the diagonal of a parallelogram of which Bc is one side, and the direction of the centripetal force at B is in the other side. Now, in the parallelogram whose diagonal is BC, and one of its sides Bc, Cc must be another; whence the opposite side, that is, the direction of the centripetal force, must be parallel to Cc: but Cc and BS have been proved to be parallel, when equal areas are described in equal times about the point S. Therefore, on this supposition, the body at B is acted upon by the centripetal force in the direction BS. The same may be shewn at every other point C, D, &c. Therefore the centripetal force tends to that point round which the body, by a radius drawn thither, describes equal areas in equal times.

## L E M. IV.

Plate 4.  
Fig. 1.

*If any arc ACB, of a finite curvature, is subtended by its chord AB, and a straight line AD produced both ways touch the arc in the point A, the arc, the chord, and the tangent, in their ultimate vanishing state, will be equal.*

The arc being supposed of a finite curvature, or such as may be measured by a circle of a finite diameter; let BAC be the circle of the curvature; draw the line AC in that circle parallel to the subtense BD, and complete the triangle BAC. Because the angle DAB (El. III. 32.) is equal to the angle ACB, and the alternate angles CAB, ABD, are equal, the triangles, CAB, DAB, are similar. Hence, (El. VI. 4.) AB is to AD, as AC is to BC. The point B approaching continually to A, let BA become less than any assignable quantity; then the finite lines AC, BC, approach nearer to the ratio of equality, than by any given distance: therefore likewise AB, AD, which are proportional to AC, BC, and much more the intermediate arc, are ultimately equal.

L E M.



## L E M. V.

*The nascent or evanescent subtense of the angle of contact, in circles and in all curves which have a finite curvature, is as the square of the conterminous arc.*

Let AD, in the semicircle ADC, be the given arc, AB the tangent, and the angle BAD the angle of contact. Draw BD, HG, parallel to AC the diameter; these lines, subtending the angle BAD, are called the subtenses of the angle of contact. The arcs AD and AG, having the common term or limit the point A, are called conterminous arcs. Draw the lines DC, GC. If these lines be conceived to turn round upon the point C as a center, so that the two points D, G, and with them the two subtenses BD, HG, may approach towards A; it is manifest, that as these subtenses came nearer to A, they will diminish, and at last will vanish in A. At the instant of their vanishing, BD will be to HG, as the square of the arc AD is to the square of the arc AG.

Plate 3.  
Fig. 17.

Let ED be drawn parallel to AB, and FG to AH. Then because AB is a tangent at the point A, and consequently (El. III. 18.) perpendicular to the diameter AC, ED which is parallel to AB is likewise perpendicular to AC: for the same reason FG is perpendicular to AC. And ADC, AGC (El. III. 31.) are right angles. Therefore (El. VI. 8.) AED is similar to ADC, and AE is to AD, as AD to AC. Therefore (El. VI. 17.) the rectangle of AE, AC, is equal to the square of AD. But AE is equal to BD: therefore the rectangle BD, AC, is equal to the square of AD. For the same reason, the rectangle of HG, AC, is equal to the square of the chord AG. Consequently, the square of the chord AD is to the square of the chord AG, as the rectangle BD, AC, is to the rectangle HG, AC, that is (El. VI. 1.) as BD to HG. But (by Lem. IV.) the arc AD and the chord AD, are ultimately in the ratio of equality, and also the arc AG, and chord AG. Therefore the square of the arc AD is to the square of the arc AG, at the instant in which they vanish, as BD to HG; that is, the evanescent subtense of the angle of contact is as the square of the conterminous arc.

Next; let the subtenses be not parallel to the diameter, but parallel to one another. Let MN, FG, be the subtenses parallel to the diameter; and AN, OG, two subtenses parallel to each other, but not to the diameter. Because BA is a tangent at the point B, BD (El. III. 16.) is perpendicular to BA: since, therefore, MN and FG are parallel to BD, they are also perpendicular to BA, and the angles OFG, AMN, are equal. But because OG and AN are parallel by construction, the angle FOG (El. I. 29.) is equal to the angle MAN. Therefore the triangles FGO, MNA, are similar, and (El. VI. 4.) AN is to OG, as MN is to FG. But it has been proved that MN is ultimately to FG, as the squares of the conterminous arcs: therefore AN is ultimately to OG, as the squares of the conterminous arcs BN, BG.

Fig. 16.

L

Lastly;



Plate 3.  
Fig. 16.

Lastly; Suppose both AN and OG directed to C, the center of the circle. In this case, each of these would be a semidiameter, continued from G and N respectively to the tangent BA. In their ultimate state these lines AN, OG, must coincide in the point B, and in the same right line BC; and therefore will become parallel, and will be, from what has been shewn, ultimately as the squares of the conterminous arcs.

Fig. 17.

If GC, DC, be beginning to move from A, they are in their nascent state; and it is manifest, that the subtenses in this state are the same, and therefore have the same ratio, as in the evanescent state.

COR. 1. Hence, because the tangents AB, AH, the arcs, AD, AG, and their fines ED, FG, became ultimately equal (by Lem. IV.) to the chords AD, AG, their squares, also will be ultimately as the subtenses BD, HG.

COR. 2. The same squares are also ultimately as those versed fines of the arcs, which bisect the chords and converge to a given point. For by the second case of this proposition these versed fines AE, AF, are as the subtenses BD, HG, or AN, OG.

COR. 3. Hence these versed fines AE, AF, are as the squares of the times in which a body describes the arcs AD, AG, with given velocities. For the spaces AD, AG, described with given velocities, are as the times, and the squares of the spaces as the squares of the times; but (by last Cor.) the squares of these spaces are as the versed fines AE, AF; therefore these versed fines are as the squares of the times in which the arcs AD, AG, are described.

### L E M. VI.

*The nascent or evanescent subtense of the angle of contact is equal to the square of the conterminous arc divided by the diameter.*

Plate 3.  
Fig. 17.

It has been shewn, in the preceding Lemma, that BD is to AD, as AD is to AC. Therefore (El. VI. 17.) BD multiplied into AC, is equal to the square of AD. Let these equal quantities be divided by AC, the quotients will be equal: but  $\frac{BD \times AC}{AC} = BD$ : therefore  $BD = \frac{AD^2}{AC}$ . But by Lem. IV. the arc AD is ultimately equal to the chord AD: therefore the nascent or evanescent subtense BD is equal to the square of the arc AD divided by the diameter AC.

### P R O P. LXXIV.

The centripetal forces of bodies revolving in different circular orbits about the same center towards which they tend, are as the squares  
of



of the arcs described in the same time, divided by the radii of the circles.

In the circular orbits BND, RLE, let bodies revolve about the center C, towards which they tend. Let them in the same time describe the indefinitely small arcs BG, RL. Then, because the projectile forces would carry them in the same time through the tangents BF, RH; and the spaces through which, at the points G and L, they have been drawn from the tangents towards the center by the centripetal force, are FG, HL; the centripetal forces must be as FG and HL. And (by Lem. VI.) the evanescent, or nascent, subtense FG is equal to the square of the arc BG divided by BD; and the evanescent, or nascent, subtense HL is equal to the square of the arc RL divided by RE. Therefore the subtense FG is to the subtense HL as the square of the arc BG divided by BD, or its half BC, is to the square of the arc RL, divided by RE or its half RC. Therefore the centripetal forces, when the arcs are nascent, are in the same ratio, that is, as the squares of the arcs divided by the radii.

Plate 3.  
Fig. 16.

And this is true, whatever arcs BG and RL be taken, if they be described in the same time: for the nascent arcs will be as the velocities; and any other arcs BND, RLE, described in any given time, will be also as the velocities; therefore, the arcs BND, RLE are as the nascent arcs BG, RL, and their squares are likewise proportional. But the centripetal forces are as the squares of the nascent arcs BG, RL, divided by the radii BC, RC; therefore these forces are as the squares of any other arcs, BND, RLE, divided by the radii of their circles.

### P R O P. LXXV.

The centripetal forces of equal bodies revolving in circular orbits, are as the squares of the velocities directly, and the radii of the orbits inversely.

Because arcs described in the same time are as the velocities, and that the centripetal forces are (by Prop. LXXIV.) as the squares of the arcs described in the same time divided by the radii, these forces are also as the squares of the velocities divided by the radii, that is, as the squares of the velocities directly, and the radii of the orbits inversely.

COR. Hence the centripetal forces of equal bodies, at equal distances from the center, are as the squares of the number of revolutions in any given time; for this number is as the velocity with which the body moves.



## P R O P. LXXVI.

The centripetal forces of equal bodies revolving in equal circular orbits are inverfely as the fquares of their periodical times.

The circular orbits or fpaces being equal, the times in which thefe are defcribed, or the *periodical times*, are (by Prop. V.) inverfely as the velocities; and therefore the fquares of the periodical times are inverfely as the fquares of the velocities, or the fquares of the velocities are inverfely as the fquares of the periodical times: but (by Prop. LXXV.) the centripetal forces are as the fquares of the velocities: therefore thefe forces are inverfely as the fquares of the periodical times.

## P R O P. LXXVII.

The centripetal forces of equal bodies revolving in unequal circular orbits, if the periodical times are equal, are as the radii of the circles.

Plate 3.  
Fig. 16.

Let one body revolve in the circular orbit BND, and another, in the fame time, in the circular orbit RLE. Becaufe the periodical times are equal, each body in any given part of its periodical time will defcribe an equal number of degrees in its refpective orbit, that is, will defcribe fimilar arcs. The arcs BN, RL, being fimilar, will be defcribed in equal portions of the periodical time: therefore (by Prop. LXXIV.) the centripetal forces will be as the fquares of the fimilar arcs BN, RL, divided by the radii BC, RC, that is, as  $\frac{BN^2}{BC}$  to  $\frac{RL^2}{RC}$ . But becaufe fimilar arcs are to each other as the circumferences, or radii, of circles, BN is to RL, as BC to RC, and confequently,  $BN^2$  to  $RL^2$ , as  $BC^2$  to  $RC^2$ : therefore  $\frac{BN^2}{BC}$  is to  $\frac{RL^2}{RC}$ , as  $\frac{BC^2}{BC}$  is to  $\frac{RC^2}{RC}$ , that is, as BC to RC.

But the centripetal forces (Prop. LXXIV.) are as  $\frac{BN^2}{BC}$  to  $\frac{RL^2}{RC}$ ; therefore thefe forces are as BC to RC, that is, as the radii of the orbits in which the bodies move.

## P R O P. LXXVIII.

The centripetal forces of equal bodies revolving in circular orbits, are as the radii of the orbits directly, and the fquares of the periodical times inverfely.

If the periodical times are equal, and the radii unequal, the forces are (by Prop. LXXVII.) as the radii. If the radii are equal, and the periodical times unequal, the forces (by Prop.



Prop. LXXVI.) are inverfely as the fquares of the periodical times. Therefore, if both the radii and periodical times are unequal, the forces will be in the compound ratio of both, or as the radii directly, and the fquares of the periodical times inverfely.

## P R O P. LXXIX.

When bodies revolve round the fame center, if the fquares of their periodical times are as the cubes of their diftances from the center, the centripetal forces will be inverfely as the fquares of their diftances.

Let the diftances of the two bodies be expreffed by  $D, d$ ; and the periodical times by  $P, p$ : then, by the fuppoſition  $P^2 : p^2 :: D^3 : d^3$ .

By Prop. LXXVIII. the centripetal forces are as the diftances directly, and the fquares of the periodical times inverfely, that is (taking  $C, c$ , for the centripetal forces)

$C : c :: \frac{D}{P^2} : \frac{d}{p^2}$ ; and by fuppoſition  $P^2 : p^2 :: D^3 : d^3$ : therefore, ſubſtituting

$D^3, d^3$ , for  $P^2, p^2$ ,  $C : c :: \frac{D}{D^3} : \frac{d}{d^3}$ , that is,  $C : c :: \frac{1}{D^2} : \frac{1}{d^2}$ ; and, be-

cauſe where the dividend is given, the quotient is inverfely as the diviſor,  $\frac{1}{D^2}$  is to  $\frac{1}{d^2}$  inverfely as  $D^2$  to  $d^2$ . Therefore  $C : c :: d^2 : D^2$ , that is, the centripetal forces are inverfely as the fquares of the diftances.

SCHOL. I. Let  $C, c$ , expreſs the central forces;  $A, a$ , the arcs deſcribed;  $V, v$ , the velocities with which the bodies move;  $P, p$ , the periodical times of their revolution;  $D, d$ , the radii or diftances from the center; and  $N, n$ , the number of revolutions in a given time;—the preceding Propoſitions may be thus expreſſed.

The bodies being equal,

PROP. LXXIV.  $C : c :: \frac{A^2}{R} : \frac{a^2}{r}$ .

LXXV.  $C : c :: \frac{V^2}{D} : \frac{v^2}{d}$ .

COR.  $C : c :: N^2 : n^2$ .

LXXVI.  $C : c :: \frac{1}{P^2} : \frac{1}{p^2}$  or  $p^2 : P^2$ .

LXXVII.  $C : c :: D : d$ .

LXXVIII.  $C : c :: \frac{D}{P^2} : \frac{d}{p^2}$

LXXIX. If  $P^2 : p^2 :: D^3 : d^3$ ,  $C : c :: \frac{1}{D^2} : \frac{1}{d^2}$  or  $d^2 : D^2$ .

SCHOL.



SCHOL. 2. Since it was proved (Prop. LXX.) that the centripetal and centrifugal forces are, in circular orbits, equal to one another, the preceding Propositions, being demonstrated respecting the centripetal force, are also true of the centrifugal force; and it may be asserted universally, that the *central forces* are in the ratios above expressed.

These Propositions may be confirmed by the following experiments, on the Whirling Tables.

EXP. 1. Let two equal balls, placed at equal distances from the center of motion on the whirling tables; and let one table revolve twice whilst the other revolves once: the ball on the table whose number of revolutions is, with respect to that of the other in the same time, as 2 to 1 (or the periodical times as 1 to 2) will raise 4 times the weight raised by the other ball; that is, according to Prop. LXXV. and Cor. the radii being equal,  $C : c :: V^2 : v^2 :: N^2 : n^2$ ; or (by Prop. LXXVI.)  $:: p^2 : P^2$ .

2. Let two equal balls be placed on tables whose number of revolutions in the same time are as 2 to 1: let the ball on the table whose number of revolutions is 2, be placed at half the distance from the center, at which the ball on the table, whose number of revolutions is 1, is placed; whence their velocities will be equal. The ball at the distance 1, will raise double the weight raised by the ball at the distance 2; that is, according to Prop. LXXV. the velocities being equal,  $C : c :: d : D$ .

3. Let two equal balls revolve, on tables whose periodical times are equal; and let the distances of the balls from the center be to each other as 2 to 1: the ball which is at the distance 2 will raise double the weight raised by the ball which is at the distance 1; that is, according to Prop. LXXVII.  $C : c :: D : d$ .

4. Let equal balls be placed on tables whose periodical times are as 2 to 1; let the ball on the table whose periodical time is 2, be placed twice as far from the center as the ball whose periodical time is 1; the ball whose distance is 2, and periodical time 2, will raise half the weight raised by the ball whose distance is 1, and periodical time 1; that is, according to Prop. LXXVIII.  $C : c :: \frac{D}{P^2} : \frac{d}{p^2} :: \frac{2}{4} : \frac{1}{2}$ .

5. Let the equal balls be so placed on different tables, that the distance of one from the center may be to that of the other as 2 to  $3\frac{1}{6}$ ; let that ball which is at the least distance revolve twice in the same time in which the other ball revolves once; the periodical time of the ball at the less distance, is to that of the ball at the greater, as 1 to 2, and the squares of the periodical times will be as 1 to 4, and the cubes of the distances are 8, and  $31.75$ : but  $1 : 4 : 8 : 32$ , therefore the squares of the periodical times being in this case nearly as the cubes of the distances, the weight raised by the ball whose distance is 2, will be to that



that raised by the ball whose distance is  $3\frac{1}{6}$  as the square of  $3\frac{1}{6}$  is to the square of 2 ; that is, nearly as 10 to 4, or 5 to 2.

P R O P. LXXX.

The centripetal forces of revolving bodies are as their quantities of matter.

For the whole centripetal force of any body is made up of the centripetal forces of each particle of matter of which it consists ; and therefore the more numerous the particles of matter in any body are, the greater will be its centripetal force.

EXP. Let two glass tubes be half filled with water ; into one put some leaden shot, and into the other a few small round pieces of light wood ; let the orifice of each tube be closed by a cork : fasten the tubes to an inclined plane, and let the lower end of it rest upon the center of the whirling table. On turning the table, the bodies will be carried by their centripetal forces from the center ; and the heavier bodies will recede farther from the center than the lighter.

COR. Hence, when the revolving bodies are not equal, the centripetal forces are in the ratios laid down in the preceding propositions multiplied into their quantities of matter. Thus  $Q, q$ , expressing the quantities of matter, and the other expressions remaining as in Prop. LXXIX. Schol.

$$C : c :: Q : q$$

$$C : c :: \frac{QV^2}{D} : \frac{qv^2}{d}$$

$$C : c :: QN^2 : qn^2$$

$$C : c :: \frac{Q}{P^2} : \frac{q}{p^2}$$

$$C : c :: QD : qd$$

$$C : c :: \frac{QD}{P^2} : \frac{qd}{p^2}$$

$$\text{If } P^2 : p^2 :: D^3 : d^3, C : c :: \frac{Q}{D^2} : \frac{q}{d^2}$$

COR. Hence the central forces will be equal, whenever the expressions proportional to them are equal ; thus,  $C = c$  if  $QD = qd$ .

Any of the above proportions may be confirmed by experiment ; for example ;

EXP. 1. Let the two balls A, B, be as 2 to 1 ; let the distance of the ball A be to that of the ball B from the center, as 2 to 1, and the periodical time of the ball A be twice that



that of the ball B; their velocities will be equal; therefore the centrifugal force of A will be to that of B, as  $\frac{Q}{D}$  is to  $\frac{q}{d}$ , that is, as 1 to 1, or A and B will raise equal weights.

2, 3. Let the same balls revolve about a fixed point, and have their distances reciprocally proportional to their quantities of matter, their centrifugal forces (compare Prop. LXXV. and LXXX.) will be equal, and they will balance each other. This may be shewn by two balls suspended freely and united by a cord, having the point of the cord which is directly above the center of the table at distances from the balls reciprocally as their weight; or by two balls united by a wire, and resting in equilibrio on a forked support fixed in the center of the tables, which will continue in equilibrio when the tables are turned.

In like manner other cases may be confirmed by experiment.

### L E M. VII.

*If a body revolves freely in any orbit about an immoveable center, and in an indefinitely small time describes any nascent arc; and the versed sine of the arc be drawn which may bisect the chord, and being produced may pass through the center of force; the centripetal force, in the middle of this arc, will be as the versed sine directly, and the square of the time inversely.*

Plate 4.  
Fig. 3.

Let two bodies revolve round their center of force S, s; let QPM, qpm, be the nascent arcs described in any times, T, t; and let PB, pb, or QR, Aa, be the versed sines bisecting the chords, and when produced, passing through S the center of force. Supposing the arcs MQP, NAp, to be described in the same time with different forces, C, c; by Prop. LXXII. Cor. 4.  $QR : Aa :: C : c$ . Hence, supposing the forces to be equal, QR is equal to Aa described in the same time; and (by Lem. V.)  $QR$  or  $Aa : qr :: Ap^2 : qp^2$ , that is, since the motion in the arcs is uniform,  $Aa : qr :: T^2 : t^2$ . Therefore supposing both the times and forces different, and compounding these ratios,

$$QR : qr :: C \times T^2 : c \times t; \text{ whence } C : c :: \frac{QR}{T^2} : \frac{qr}{t^2}$$

Plate 4.  
Fig. 4.

COR. I. If a body P, revolving about the center S, describes a curve line APQ, and a right line ZPR touches that curve in any point P; and, from any other point Q of the curve, QR is drawn parallel to the distance SP, meeting the tangent in R; and QT is drawn perpendicular to the distance SP; the centripetal force will be reciprocally as the quantity  $\frac{SP^2 \times QT^2}{QR}$ , if this be taken of that magnitude which it ultimately acquires, supposing the points P and Q continually to approach to each other. For QR is equal

to



to the versed sine of double the arc  $QP$ , in whose middle is  $P$ : and double the triangle  $SQP$ , or  $SP \times QP$  (or ultimately  $QT$ ) is proportional to the time, in which that double arc is described (by Prop. LXXII.) and therefore may be used for the exponent of the time.

Whence  $C : c :: \frac{QR}{SP^2 \times QT^2} : \frac{qr}{sp^2 \times qt^2}$ ; that is,  $C$  is to  $c$  reciprocally as  $\frac{SP^2 \times QT^2}{QR} : \frac{sp^2 \times qt^2}{qr}$ ; or the centripetal forces are reciprocally as  $\frac{SP^2 \times QT^2}{QR}$

COR. 2. Hence, if any curvilinear figure  $APQ$  is given; and therein a point  $S$  is also given, to which a centripetal force is perpetually directed; the law of centripetal force may be found, by which the body  $P$ , continually drawn back from a rectilinear course, will be retained in the perimeter of that figure, and will describe the same by a perpetual revolution. That is, we are to find the quantity  $\frac{SP^2 \times QT^2}{QR}$ , reciprocally proportional to this force.

P R O P. LXXXI.

If equal bodies, revolving in ellipses, describe equal areas in equal times, their centripetal forces are to one another inversely as the squares of their distances from the *foci* of the ellipse towards which they tend.

Let  $S$  be the focus: let a body  $P$ , tending towards  $S$ , describe a part of the ellipse  $PQ$ ; join  $SP$ ; draw  $QR$  to the tangent  $YZ$ , parallel to  $SP$ ; join  $PC$ , and produce it to  $G$ . Complete the parallelogram  $QxPR$ , produce  $Qx$  to  $v$ ,  $Qv$  is ordinately applied to  $GP$ ; draw  $DK$ , a diameter parallel to  $YZ$ , and draw  $IH$  from the other focus  $H$  to  $SP$  parallel to  $YZ$ ; join  $HP$ , and draw  $QT$  perpendicular to  $SP$ , as also  $PF$  to  $DK$ .

Plate 4.  
Fig. 4.

$EP$  is equal to the greater semiaxis  $AC$ . For, because  $CS$  is equal to  $CH$ ,  $ES$  is equal to  $EL$ , whence  $EP$  is half the sum of  $PS$ ,  $PI$ , that is, of  $PS$ ,  $PH$ , (for El. I. 13.  $RPI$ ,  $IPZ$ , are equal to  $HPR$ ,  $HPZ$ ; and taking away the common part  $HPI$ , the angle  $IPR$  is equal to  $HPZ$ ; whence (El. I. 29.) the angle  $PIH$  is equal to  $PHI$ , and  $PI$  is equal to  $PH$ ); and  $PS$ ,  $PH$ , together (Simson's Conic Sect. II. 1.) are equal to the whole axis  $2AC$ .  $EP$  therefore is equal to  $AC$ .

Putting  $L$  for the principal *latus rectum* of the ellipse,  $L$ , (by definition) is equal to  $\frac{2BC^2}{AC}$  (for  $AC : CB :: CB : \frac{L}{2}$ , whence  $\frac{CB^2}{AC} = \frac{L}{2}$ , or  $\frac{2CB^2}{AC} = L$ ). And  $L \times QR : L \times Pv :: QR : Pv$ ; and  $QR = Px$ ; and  $Px : Pv :: PE : PC$ ; whence  $L \times QR : L \times Pv :: PE$  or  $AC : PC$ . And (El. VI. 1.)  $L \times Pv : Gv \times Pv :: L : Gv$ ; and (Simf. II. 15.)  $Gv \times Pv : Qv^2 :: PC^2 : DC^2$ . And (Lem. IV.) the points  $Q$  and  $P$  continually approaching,

M



proaching,  $Qv^2$  is to  $Qx^2$  ultimately in the ratio of equality. And (since the triangles  $QTx$ ,  $EPF$ , are similar, for  $QxT = PEF$ , and  $QTx$  to  $EFP$ )  $Qx^2$  or  $Qv^2 : QT^2 :: EP^2$  or  $AC^2 : PF^2$ . But because (Simf. II. 20. Compare Vince's Con. Sect. II. 10. Cor. 1.) parallelograms about conjugate diameters are equal to the rectangle under the axes, the rectangle  $PF$ ,  $DC$ , is equal to the rectangle  $ACB$ , whence  $PF : AC :: CB : DC$ , and  $AC^2 : PF^2 :: CD^2 : CB^2$ , wherefore  $Qv^2 : QT^2 :: CD^2 : CB^2$ . Compounding the following ratios,

$$\begin{aligned} L \times QR &: L \times Pv :: AC : PC, \\ L \times Pv &: Gv \times Pv :: L : Gv, \\ Gv \times Pv &: Qv^2 :: PC^2 : CD^2, \\ Qv^2 &: QT^2 :: CD^2 : CB^2; \end{aligned}$$

And, striking out the equal quantities,  $L \times QR : QT^2 :: AC \times L \times PC : Gv \times CB^2$ .

Then substitute for  $AC \times L$  its equal  $2CB^2$ , and

$$\begin{aligned} L \times QR : QT^2 &:: 2BC^2 \times PC : Gv + BC^2 \\ &\text{or } BC^2 \times 2PC : Gv \times BC^2 \\ &\text{or } 2PC : Gv \end{aligned}$$

But the points  $Q$  and  $P$  continually approaching without end,  $2PC$  and  $Gv$  are equal : wherefore  $L \times QR$  and  $QT^2$ , proportional to these are also equal. Multiply these equals into  $\frac{SP^2}{QR}$  and  $L \times SP^2$  will become equal to  $\frac{SP^2 \times QT^2}{QR}$ .

Therefore (by Lem. VII. Cor. 1 and 2.) the centripetal force is reciprocally as  $L \times SP^2$ , that is, since  $L$  is a given quantity, as  $SP^2$ , or in a duplicate ratio of the distance  $SP$ .



B O O K III.

OF HYDROSTATICS AND PNEUMATICS;

OR THE LAWS OF

INCOMPRESSIBLE AND COMPRESSIBLE FLUIDS.

P A R T I.

OF HYDROSTATICS.

C H A P. I.

*Of the Weight and Pressure of Fluids.*

DEF. I. **A** FLUID is a body, the parts of which yield to any force impressed upon them, and easily move out of their places.

PROPOSITION I.

The weight of fluids is as their quantities of matter.

Since each particle of any fluid gravitates towards the earth, the greater is the number of particles, that is the quantity of matter, in any mass of fluid, the greater will be the weight of that mass.



EXP. I. The different pressures of different columns of fluid in the same vessel at different depths, appear from the different quantities of fluid discharged, at different depths, in the same time, from orifices of the same bore.

2. If the air be exhausted from a tube in part filled with water, and the tube be closed up, the solidity of the particles of water will be perceived from the sound produced by suddenly lifting up the tube.

COR. Fluids gravitate in fluids of the same kind. For they cannot lose the property of gravity which belongs to all bodies by such a change of situation.

EXP. Suspend a stopped phial from one arm of a balance, in a vessel of water, and balance it by weights from the opposite arm of the balance: upon unstopping the phial under water, a quantity of water will rush into it, by which the weight will be increased as much as the weight of the water in the phial.

### P R O P. II.

When the surface of a fluid is level, the whole mass will be at rest.

Plate 5.  
Fig. 2.

Let ABCD be a vessel containing water, the level surface of which is EF. Conceive the whole mass of fluid in the vessel to be divided into thin *strata*, or plates, RS, TV, XY, &c. lying horizontally one above another; and into small perpendicular columns GH, IK, LM, &c. contiguous to each other. In the stratum XY, and the columns IK, let  $m$ ,  $n$ , be two adjacent drops. Neither of these drops can move towards the column in which the other is, without driving that other out of its place, because the fluid is supposed incompressible. But, with whatever force the particle  $m$  endeavours to displace the particle  $n$ , this force is counterbalanced by an equal and contrary effort on the part of  $n$ ; because (Prop. I.) they are equally pressed by the equal columns above them: consequently the particles will be at rest.

### P R O P. III.

Any part of a fluid at rest presses, and is pressed, equally in all directions.

For (Def. I.) each particle is disposed to give way on the slightest difference of pressure: and, by supposition, each particle is pressed by the contiguous particles in such manner as to be kept at rest in its place: it is therefore pressed with an equal degree of force on all sides; and consequently (Book. II. Prop. III.) it presses equally in all directions.

EXP. I. Into several tubes, bent near their lower ends in various angles, pour a sufficient quantity of mercury to fill the lower parts of their orifices; then dip them into  
a deep



a deep glass vessel filled with water, keeping the orifice of the longer legs above the surface: whilst the tubes are descending, the mercury will be gradually pressed upwards in the tubes, and the pressure will be equal at any given depth, whatever be the direction of the pressing column of fluid in the shorter leg of the tube. Oil may be used instead of mercury.

2. Dip an open end of a tube, having a very narrow bore, into a vessel of quicksilver; then, stopping the upper orifice with the finger, lift up the tube out of the vessel: a short column of quicksilver will hang in the lower end, which, when dipped in water lower than 14 times its own length, will, upon removing the finger, be suspended, and pressed upwards.

3. Let a large open tube be covered at one end with a piece of bladder drawn tight: pour into the tube a quantity of coloured water sufficient to press the bladder into a convex form; then, dip the covered end of the tube slowly into a deep vessel of water; the bladder, by the upward pressure, will become first less convex, then plane, and at last concave.

4. If the like be done with several tubes, whose covered orifices are cut obliquely at different angles, the lateral pressure will be seen to increase with the depth to which the tubes are immersed.

5. Let a circular piece of brass, whose upper surface is covered with wet leather, be held close to one orifice of a large open tube, by means of a cord or wire fastened to the middle of the plate and passing through the tube: let the plate, thus kept close to the orifice of the tube, be immersed with the tube into a large vessel of water: when the plate is at a greater depth than 8 times its thickness in the water, the cord or wire may be left at liberty, and the upward pressure of the fluid will keep the plate close to the tube.

6. Let a small bladder, tied closely about one end of an open tube having a large bore, be filled with coloured water till the water rises above the neck of the bladder; upon immersing the bladder into a vessel of water, the bladder will be compressed on all sides, and the coloured water will be raised up in the tube in proportion to the depth to which the bladder is sunk.

#### P R O P. IV.

When a fluid flows through a tube which is wider in some parts than in others, the velocity of the fluid will, in every section of the tube, be inversely as the area of the section.

Let ADMN, a bended tube larger at IL than at FG, be filled with water to the height ADFG. Let the water be forced downwards in the part ADBP, and consequently be made to rise in the other part KHMN. It is manifest, that the water which is forced out of one part of the tube, is driven into the other. Hence equal quantities pass through every section of the tube in the same time: for if less, or more, water passed through the section

Plate 5,  
Fig. 1.



section FG than through IL in the same time, the quantity of water between FG and IL must be increased or diminished, which cannot be, since no cause is supposed which could increase or diminish it. But if equal quantities pass through unequal parts of the tube in the same time, the water must run proportionally faster where the tube is narrower, and slower where it is wider. If, for example, as much water runs through the section FG, as runs in the same time through the section IL, the water must move as much faster at FG than it moves at IL, as the tube is narrower at FG than at IL; that is, the velocity is inversely as the area of the section.

COR. The momentum will be the same in every section of the tube: for the quantity of water at each section is directly as the area of the section, and the velocity is inversely as the area; therefore the velocity is inversely as the quantity of matter: whence (Book II. Prop. XI.) the momentum is every where the same.

SCHOL. Hence we may account for the suspension of the fluid in a tube the upper part of whose bore is capillary, and the lower of a much larger dimension, as was seen in the experiment, Book I. Prop. VII.

Plate 5.  
Fig. 3.

Let there be a tube consisting of two parts DR and RCK, of different diameters; DR the smaller part of the tube, is able (Book I. Prop. VIII.) to raise water higher than the other: let then the height to which the larger would raise it, be TC, and that to which it would rise in the lesser, if continued down to the surface of the fluid, be XH. If this compound tube be filled with water, and the larger orifice CK be immersed in the same fluid, the surface of the water will sink no farther than XL, the height to which the lesser part of the tube would have raised it. But if the tube be inverted, and the smaller orifice XL be immersed, the water will run out till the surface falls to TF; the height to which the larger part of the tube would have raised it.

Fig. 4.

Fig. 3.

Let the tube DR be conceived to be continued down to HI; and let it be supposed that the fluids contained in the tube XLHI, and the compound one XLCK, are not suspended by the ring of glass at XL, but that they press upon their respective bases, HI and CK. Let it farther be supposed that these bases are each of them moveable, and that they are raised up or let down with equal velocities; then will the velocity with which XL the uppermost stratum of the fluid XLCK moves, exceed that of the same stratum, considered as the uppermost of the fluid in the tube XLHI, as much as the tube RCK is wider than DR (by this Prop.) that is, as much as the space MNKC exceeds XLIH. Consequently, the effect of the attracting ring XL, as it acts upon the fluid contained in the vessel XLCK, exceeds its effect, as it acts upon that in XLHI, in the same ratio. Since therefore it is able to sustain the weight of the fluid XLHI by its natural power, it is able under this *mechanical* advantage, to sustain the weight of as much as would fill the space MNKC: but the pressure of the fluid XLCK is equal to that weight, as having the same base and an equal height (as will be shewn Prop. VI.) Its pressure therefore, or the tendency it has to descend in the tube, is equivalent to the power of the attracting ring XL, for which reason it ought to be suspended by it.

Again,



Again, the height at which the attracting ring in the larger part of the tube is able to sustain the fluid is no greater than NF, that to which it would have raised it, had the tube been continued down to MN. For here the power of the attracting ring acts under a like *mechanical* disadvantage, and is thereby diminished as much as the capacity of the tube TFMN is less than that of HIXL; because, if the bases of these tubes are supposed to be moved with equal velocities, the rise or fall of the surface of the fluid TFXL, would be so much less than that of TFMN. And, since the attracting ring TF is able by its natural power to suspend the fluid only to the height NF in the tube TFMN; it is in this case able to sustain no greater pressure than what is equal to the weight of the fluid in the space HIXL: but the pressure of the fluid TFXL, which has equal height, and the same base with it, is equal to that weight; and therefore is a balance to the attracting power.

Fig. 4.

From hence we may clearly see the reason, why a small quantity of water put into a capillary tube, which is of a conical form, and laid in an horizontal situation, will run towards the narrower end. For let AB be the tube, and CD a column of water contained within it; when the fluid moves, the velocity of the end D will be to that of the end C reciprocally as the cavity of the tube at D, to that at C (by this Prop.) that is, (El. XII. 2.) reciprocally as the square of the diameter at D, to the square of the diameter at C; but the attracting ring at D is to that at C, singly as the diameter at D to the diameter at C. Now, since the effect of the attraction depends, as much upon the velocity of that part of the fluid where it acts, as upon its natural force, its effect at D will be greater than at C; for though the attraction at D be less in itself than at C, yet its loss of force upon that account, is more than compensated by the *mechanical* advantage it has arising from hence, that the velocity of the fluid in that part is more increased than the force itself is diminished at C. The fluid will therefore move towards B.

Plate 5.  
Fig. 5.

## P R O P. V.

In bended cylindrical tubes, fluids at rest will be at the same height on each side.

In the tube ADMN, filled with water to the height AD, the water cannot descend from AD, without rising towards MN. The water in each side of the vessel may therefore be considered as two forces acting upon each other in contrary directions: and consequently these two masses of fluid will only be at rest when their momenta are equal, that is, (Book II. Prop. XI. Cor.) when the quantities of matter are inversely as the velocities, or (Prop. IV.) directly as the area of the section through which it flows. Thus, at the sections BP, KH, the momenta are equal, when the quantities of matter, or cylindrical masses of fluid are as the areas of the sections, that is, as the bases of the cylinders ADBP, FGHK. But cylinders are as their bases (El. XII. 11.) only when their

Plate 5.  
Fig. 1.



their perpendicular heights are equal. Therefore the momenta of the two cylinders of fluid will be equal, and consequently the mass will be at rest, only when the perpendicular heights of each column are equal.

EXP. 1. In a bended tube of large but unequal bore, water will rise to the same height on each side.

2. Let water spout upward through a pipe, having a small orifice, inserted into the bottom of a deep vessel; it will rise nearly to the height of the upper surface of the water in the vessel. The resistance of the air, and of the falling drops, prevents it from rising perfectly to the level.

SCHOL. In this demonstration, we do not consider the velocity with which the two columns of fluid are moving, but the velocity with which, if they move at all, they must begin to move. And since, if their perpendicular height is the same, the velocity with which they must begin to move will be inversely as their respective quantities of matter, they cannot begin to move but with equal momenta; and their motions must be in contrary directions, because one column cannot descend without making the other ascend: therefore those equal momenta would destroy each other. These two columns then, making a continual effort to move with equal momenta in contrary directions, counter-balance each other.

#### P R O P. VI.

The pressure of fluids is proportional to the base, and the perpendicular height of the fluid, whatever be the form of the vessel or quantity of the fluid.

Case 1. Let the fluid be contained in a perpendicular cylindrical vessel.

Plate 5.  
Fig. 2.

In such a vessel, ABCD, because the whole weight of the fluid, and no other force, presses directly upon the bottom CD, the pressure (by Prop. I.) must be as the quantity, that is (El. XII. 11, 14.) as the base and perpendicular height of the fluid.

Case. 2. Let the fluid be contained in a perpendicular vessel, the bottom of which is equal to that of the cylinder in the last case, but its top narrower than the bottom.

Plate 5.  
Fig. 6.

Let the vessel DBLP, have the portions of its base LA, CP, each equal to OR. From Prop. I. and III. it appears, that each of these portions are equally pressed by the column DBOR, as the base OR. In like manner, every portion of the base LP equal to OR is as much pressed as OR. Therefore the whole base LP is as much pressed as if the vessel was of the cylindrical form FHLP.

Or thus: Because (by Prop. V.) if a tube were inserted at NT, of the diameter OR, the water being at the height DB, would rise to the level FE, there must at NT, be an upward pressure towards F sufficient to fill up the columns of fluid FELA, that is, equal to



to the weight of as much water as would fill the space FENT. Consequently the re-action, that is, the pressure upon the base LA must be equal to the weight of as much water as would fill FENT. But the base LA supports this re-action, and likewise the weight of the water NTLA, which are together equal to the weight of DBOR. The base LA therefore sustains a pressure equal to the weight of the column DBOR. And every equal portion of the base may, in the same manner, be shewn to sustain an equal pressure. Therefore, the pressure on the base is the same in vessels of the form supposed in this case, as in cylinders of equal bases, and of the same altitude with these vessels. The same may be shewn with respect to a vessel of the form of plate 5, fig. 7.

Case. 3. Let the vessel be of the same base and altitude, but have its top wider than the base.

Let the fluid of the vessel be divided into strata EF, GH, IK, &c. Let us also imagine the bottom of the vessel C to be moveable, that is, capable of sliding up and down the narrow part of the vessel, from C to GH. Let it further be supposed that this moveable bottom, is drawn up or let down with a given velocity, while the vessel itself is fixed and immovable; it is evident the lowest stratum, which is contiguous to the bottom, will be raised or let down with the same velocity, and will therefore have a momentum proportional to that velocity, and the quantity of matter it contains: but (by Prop. IV. Cor.) the rest of the strata will have the same momentum: consequently, the momentum of all taken together, that is, of the whole fluid, is the same as if the vessel had been no larger in any one part than it is at the bottom, for then the momentum of each stratum would also have been as great as that of the lowest. The pressure therefore, or action of the fluid, with which it endeavours to force the bottom out of its place, is as the number of strata, that is, the perpendicular height of the fluid, and the magnitude of the lowest stratum, that is, the base.

Plate 5.  
Fig. 8.

Case 4. Let the fluid be in an inclined cylindrical vessel.

In the inclined cylindrical vessel ABNI, as much as the fluid is prevented from pressing upon the base NI, by being in part supported by the side of the vessel AN, so far is the pressure upon the base increased by the re-action of the opposite side BI, which is equal to the action of the former, because the fluid, pressing every way alike at the same depth below the surface, exerts an equal force against both the sides. The base NI is therefore pressed with the same force with which it would be pressed, if the fluid contained in the vessel ABNI was included in the vessel EDIO, having an equal base, and the same perpendicular height with the vessel ABNI; that is (by the first case) the pressure is as the base NI and altitude CN.

Plate 5.  
Fig. 9.

Since then, the pressure upon the base of vessels, either wider or narrower at the top than the bottom, and likewise the pressure upon the base of vessels inclined to the horizon, is equal to that upon the base of a cylindrical vessel of the same base and height, the sides of which are perpendicular to the horizon; and since the pressure upon the base of such

N

a cylinder



a cylinder is as its base and height; the pressure upon the bottom of all vessels filled with fluid, is proportional to their base and perpendicular height.

EXP. 1. Let two tubes of different forms be successively applied to the same moveable circular base, suspended by a wire, passing from the center of the base through the tubes, to the beam of a balance: when the different tubes are filled to the same height, it will require the same weight at the opposite end of the balance to keep the base from sinking.

2. Let two tubes, the one cylindrical, the other of the form of a speaking trumpet, have their bases of equal diameter, covered with bladder and inserted in a vessel of water, as in Prop. III. Exp. 3. the bladder will become plane at the same depth in both: from whence it appears, that since the upward pressures, at the same depth, are equal, the downward pressures in the two tubes are also equal.

COR. 1. A small quantity of fluid may be made to press with a force sufficient to raise a great weight.

Plate 5.  
Fig. 6.

Since (as was shewn in Prop. V.) as much fluid as will fill the tube DBIV presses upwards against VM, with a force equal to the weight of as much fluid as would fill the space BHVM; the base remaining the same, the space BHVM, that is, the weight which may be raised will (by this Prop.) be as the height VB, which may be increased at pleasure.

EXP. Let two circular pieces of wood be united by leather in the manner of a pair of bellows; in the upper board insert a long tube with a large bore; through which pour water into the vessel; the upward pressure of the water as it is poured in, will raise a great weight.

COR. 2. From hence it may be proved, independently of the reasoning in Prop. V. that, in bended vessels or channels of any form, fluids rise to the same height, whatever be the difference between the quantities of fluid on each side: for whatever be the form of the channels, the plane which is perpendicular to the lowest point being considered as the common base, the pressure upon it, is equal, when the fluid on each side is of equal altitude; and the whole mass can only be at rest when the opposite pressures are equal.

SCHOL. This pressure of the fluid upon the base does not alter the weight of the vessel and fluid considered as one mass, because the action and re-action which cause it, with respect to the weight of the vessel, destroy one another; the vessel being as much sustained by the action upwards, as it is pressed by the re-action downwards.

PROP.



## P R O P. VII.

The pressure of a fluid upon any indefinitely small part of the side of the vessel which contains it, is equal to the weight of a column of the same fluid, whose base is the part pressed, and whose height is the distance of that part from the surface of the fluid.

Let ABCD be a vessel filled with fluid; AB its surface; and L a point in the side of the vessel. The indefinitely small drop which lies next to the point L is pressed downwards (by Prop. I.) by a force equal to the weight of a column of water whose base is L, and height LA, the distance of that part from the surface. And (by Prop. III.) this drop is pressed sideways towards L with the same force with which it is pressed downwards. Whence the proposition is manifest concerning the point L. And the same may be proved concerning any other points M, N, C, equal to L. The same is evidently true in an inclined vessel.

Plate 3.  
Fig. 12.

## P R O P. VIII.

The pressure of a fluid upon any plane, is equal to the weight of a body which has the same density with the fluid, and is formed by raising perpendiculars upon each indefinitely small part of the plane, equal in height to the distance of that part from the surface of the fluid.

It has been proved, in the last proposition, that the pressure upon each indefinitely small part of the line AC in the side of the vessel ABCD is equal to the weight of a column of fluid whose base is the part pressed, and whose height is the distance of that part from the surface AB. Hence, if from the point L a perpendicular LO be raised whose base is L, and whose length LO is equal to LA the distance of L from the surface, if this perpendicular consisted of matter of the same density with the fluid in the vessel, the weight of this perpendicular column would be equal to the pressure upon the point L. If, in like manner, perpendiculars, consisting of matter of the same density with the fluid, were raised upon every point between A, C, they would together fill up the area of the triangle ACD; and the pressure upon the whole line AC in the side of the vessel ABCD, because it is equal to the sum of the pressures upon all its parts, must be equal to the weight of this triangle ACD. The same may be proved concerning any other lines in the side of the vessel, as HI, EF. Consequently, the pressures upon the whole side will be equal to the weight of as many such triangles as there can be lines drawn upon it

Plate 5.  
Fig. 12.

Fig. 14.



in the same manner as  $AC$ ,  $HI$ ,  $EF$  are drawn. But all these triangles together would fill up the whole space, or compose a solid,  $CFGDAE$ . Therefore the pressure upon the side  $AECF$  will be equal to the weight of this solid consisting of matter which has the same density with the fluid in the vessel; which solid is formed by raising perpendiculars upon each line of the side, respectively equal to the distance of that point from the surface of the fluid.

Plate 5.  
Fig. 13.

In like manner, if  $AC$  is a line drawn in the inclined side of a vessel, in which the water reaches to the level  $AB$ , the pressure upon this line may be estimated as before.  $SL$  is the distance of  $L$  from the surface. Let therefore a perpendicular  $LO$  equal in length to  $LS$  be raised upon the point  $L$ ; then, if this perpendicular was a column of matter of the same density with water, the weight of it would be equal to the pressure upon  $L$ . For the same reason, if a perpendicular  $MP$  is raised upon the point  $M$ , and is made equal in length to  $MT$  the distance of  $M$  from the surface; such a perpendicular, consisting of matter of the same density with water, and being of the same size, would have the same weight as the column of water  $MT$ . And since (by Prop. I.) the pressure upon  $M$  equals the weight of the incumbent water  $MT$ , it likewise equals the weight of the perpendicular  $MP$ . In like manner, the points  $N$  and  $C$  are pressed by the weight of the incumbent columns  $NV$  and  $XC$ , which is equal to the weight of the perpendiculars  $NQ$ ,  $CR$ , supposing those perpendiculars to be equal in height to  $NV$  and  $XC$ , and to consist of matter whose density is the same with that of the columns  $NV$  and  $XC$ . Thus the pressure upon the whole line, being made up of the pressures upon all its parts, will be equal to the weight of as many perpendiculars, as can be raised in this manner between  $A$  and  $C$ . The sum of all those perpendiculars is the triangle  $ACR$ , whose weight therefore is equal to the pressure upon the line  $AC$ . But if as many such triangles were added together, as there are lines parallel to  $AC$  in the whole side of the vessel, all these triangles together would form a solid. And since this solid is the sum of all the pressures upon each point of the side, the weight of it, supposing it to consist of matter that has the same density as water, would be equal to the pressure upon the whole side.

### P R O P. IX.

The pressure upon any one side of a cubical vessel filled with fluid, is half the pressure upon the bottom.

Plate 5.  
Fig. 14.

The bottom sustains a pressure equal to the whole weight of the fluid in the vessel. And the pressure which the side sustains is equal to the weight of the prism  $CFGDAE$ , which (El. XI. 28.) is half the cube: therefore the side sustains a pressure equal to half the pressure upon the bottom.

Or



Or thus : Because the pressure upon every part of the vessel at the bottom is equal to the weight of a column whose base is the part pressed upon, and height that of a perpendicular from the bottom to the surface ; if the pressure were the same every where from the top to the bottom, it would be equal to the weight of as many such columns as would correspond to all the parts of the vessel. But the pressure every where diminishes as we approach towards the surface, where it is nothing ; the pressure on the side is therefore only half of that on the bottom of the vessel : a number of terms in arithmetical progression beginning from nothing being half the sum of an equal number of terms, each of which is equal to the last in the progression.

COR. 1. The gravity of the fluid in a cubical vessel producing upon each of the four sides a pressure equal to half that upon the bottom, and upon the bottom a pressure equal to itself, produces on the whole a pressure three times as great as itself.

COR. 2. When the area of the part pressed is given, the pressure is as the perpendicular distance of that part from the surface : where the depth of the part is given, the pressure is as the area ; and where the distance from the surface is given, the pressure is as the base.

## C H A P. II.

### *Of the Motion of Fluids.*

#### S E C T. I.

#### *Of Fluids passing through the Bottom or Side of a Vessel.*

#### P R O P. X.

The momentum with which any fluid runs out of a given orifice in the bottom or side of a vessel, is proportional to the perpendicular depth of the orifice below the surface of the fluid.

The pressure of a fluid against any given surface being (by Prop. I. and III.) proportional to the perpendicular height of the fluid above that part ; if that given surface be removed, the fluid will be driven through the orifice by this pressure. The force therefore with which the fluid passes through the orifice is as the perpendicular depth of the orifice  
below



below the surface of the fluid: but the momentum is always as the moving force: therefore the momentum is also as the perpendicular depth of the orifice.

## P R O P. XI.

The momentum with which any fluid runs out of a given orifice in the bottom or side of a vessel, is as the square of its velocity, and also as the square of the quantity of matter.

The momentum (by Book II. Prop. XIV.) is in the compound ratio of the quantity of matter and velocity. And it is manifest, that, since the orifice is given, the quantity of fluid discharged will always be as the velocity: therefore the momentum is as the square of the velocity, or of the quantity of fluid.

## P R O P. XII.

The velocity with which any fluid runs out of an orifice in the bottom or side of a vessel, is as the square root of the perpendicular depth of the orifice from the surface of the fluid.

Because the momentum is as the square of the velocity (by Prop. XI.) and as the perpendicular depth of the orifice (by Prop. X.) the square of the velocities (El. V. 11.) is as the perpendicular depth, and consequently, the velocity as the square root of the perpendicular depth.

## P R O P. XIII.

A fluid runs out of an orifice in the bottom or side of a vessel, with the velocity which a heavy body would acquire in falling freely through a space, equal to the perpendicular distance of the orifice from the surface of the fluid.

Plate 5.  
Fig. 10.

Let ABCD be a vessel filled with any fluid, to the height FG. It is manifest, that at the beginning of the fall of each drop from the upper surface FG, it must be carried downward by its gravity with the same velocity with which any other heavy body would begin to descend. And, if an orifice be made in the vessel at L, any point below the surface, the fluid which passes through that orifice, will (by Prop. XII.) move with a velocity which is as the square root of the distance from the surface. But if a body were to fall from the surface to the point L, it would acquire a velocity which would be



be (by Book II. Prop. XXVI. Cors.) as the square root of this distance. Therefore, since the velocity with which the fluid moves, is, at the beginning of its motion, equal to that of a falling body, and since at every given distance, these velocities have the same ratio, namely, that of the square root of the distance from the surface, that is (El. V. 9.) are equal, the proposition is manifest.

## P R O P. XIV.

When two cylindrical vessels have their bases, heights and orifices equal, if one of them be always kept full, it will discharge double the quantity of fluid discharged in the same time by the other whilst it empties itself.

For (by Prop. I.) the fluid will continue through the whole time, to run with the same velocity out of the vessel that is kept full. But the fluid will run (compare Prop. I. and XII.) with an uniformly retarded velocity out of the vessel which empties itself. And, since both vessels are full at first, the velocity which continues uniform in one vessel, will (by Prop. I.) be the same with the first velocity in the vessel in which the fluid is uniformly retarded. Therefore the quantity discharged out of the former vessel, will be to the quantity discharged in an equal time out of the latter, as the space described by a body moving with an uniform velocity, to the space described by a body which sets out with the same velocity, and is uniformly retarded. But (by Book II. Prop. XXVII.) the space described by the former will be double of the space described by the latter. Therefore the quantity discharged out of the former vessel, will be double of the quantity discharged out of the latter.

EXP. Let two vessels be discharged together, or the same vessel twice successively, according to the proposition.

## P R O P. XV.

A stream of any fluid which spouts obliquely forms a parabola.

Each drop in a stream of fluid spouting obliquely, is a heavy body projected obliquely by the force or pressure which drives it out of the orifice. Therefore (by Book II. Prop. LVIII.) every drop of the stream, that is the whole stream, forms a parabola.

EXP. Observe the figure formed by a fluid spouting obliquely.

COR. Hence fluids spouting obliquely are subject to the laws of projectiles laid down Book II. Ch. VII. Sect. I.

P R O P.



## P. R O P. XVI.

When a fluid spouts horizontally from an orifice in the side of a vessel which is kept full, if the side of the vessel is made the diameter of a circle, and a line is drawn perpendicular to the side from the orifice to the circumference, the distance to which the fluid will spout will be double of this perpendicular.

Plate 5.  
Fig. 11.

Let AB be the side of the vessel; C, E, or *e*, the orifice; ADHB the ~~line~~ <sup>semicircle</sup> drawn on the side, ED, CH, *de*, lines drawn perpendicular to the side from the orifice to the circumference. The fluid spouts at E (by Prop. XIII.) with the velocity which a heavy body would acquire in falling from A to E; and this motion, being in an horizontal direction, can neither be accelerated nor retarded by the force of gravitation, and will therefore continue uniform. But besides this, the fluid spouts with the velocity which it acquires in falling after it has passed the orifice. This velocity, when the fluid arrives at GB, is the same with that which any other heavy body would have acquired in falling through an equal space from E to B. Let this velocity be called the descending velocity, and that with which the fluid spouts at E the horizontal velocity. Then, since the horizontal velocity is the same with that which a body would acquire by falling from A to E, and the descending velocity when the fluid arrives at the plane GB, is the same with that which a body would acquire by falling from E to B, and since (by Book II. Prop. XXVI.) the spaces AE, EB, described by falling bodies are as the squares of the last acquired velocities of bodies falling through them; that is, (inverting the terms) the squares of these last acquired velocities, or the squares of the horizontal and descending velocities, are as the lines AE, EB. But in the triangle ADB, right-angled (El. VI. 8.) at D, DE is a mean proportional between AE, EB, and the square of AE is to the square of ED as AE is to EB. But the square of the horizontal velocity is to the square of the last descending velocity as AE to EB. Therefore the square of the horizontal velocity is to the square of the last descending velocity as the square of AE to the square of ED: whence the horizontal velocity is to the last descending velocity as AE to ED. Now the spaces described in the same time, in uniform motions, are (Book II. Prop. VI.) as the velocities. Consequently, if the fluid had begun to fall from E with the velocity it has acquired at B, and had fallen uniformly, in the time of descent the spaces described by the horizontal and descending velocities would have been respectively as those velocities, that is, as AE to ED. Thus while the fluid was descending till it reached the plane GB, the horizontal velocity would have carried it forward through a space equal to ED, or the horizontal distance would be ED. But the descending velocity being at the first nothing, and continually increasing, the time of descent (see Prop. IX.) is twice what it would have been upon the supposition that it began to descend with the last



last acquired velocity. And the horizontal velocity is uniform, and therefore in twice the time, or the true time of descent, the fluid will be carried horizontally to twice the distance ED. Consequently, if BF be made equal to twice DE, whilst the stream is descending from E to GB it will be carried forwards to the point F. The same may be proved concerning any other points C, *e*.

Exp. Let water spout horizontally from an orifice, the distance to which it will be projected (allowing for the resistance of the air) will agree with the proposition.

## P R O P. XVII.

If a fluid spouts horizontally out of orifices in the side of a vessel which is kept full, it will spout to the greatest distance from the orifice which is in the middle of the side, and to equal distances from orifices equally distant from the middle.

Let C be the orifice in the middle of the side, and E, *e*, equal orifices at equal distances from C.

Plate 5.  
Fig. 11.

The distance to which the fluid will spout at C (by Prop. XVI.) is twice CH, and at E twice ED. But CH (El. III. 15.) is greater than DE, any line drawn from the diameter parallel to the radius: therefore twice CH is greater than twice ED.

Also, since the horizontal distances to which the fluid will spout at E and *e*, are twice ED, or *ed*; and that ED, *ed*, being equally distant from the center and parallel to the radius (El. III. 14.) are equal; the horizontal distances from E, *e*, are equal.

Hence if, in the plane of the horizon, GB be drawn perpendicular to the side AB, and GB be double of CH, and FB double of DE, or *de*, the fluid spouting from C will fall upon G, and from E and *e*, upon F.

Exp. Let water spout from the middle orifice, and from orifices equally distant from the middle, the truth of the proposition will be manifest.

DEF. II. A *river* is a stream of water which runs by its own weight down the inclined bottom of an open channel.

DEF. III. A *section of a river* is an imaginary plane, cutting the stream, which is perpendicular to the bottom.

DEF. IV. A river is said to *flow uniformly* when it runs in such a manner, that the depth of the water in any one part continues always the same.



## P R O P. XVIII.

If a river flows uniformly, the same quantity of water passes in an equal time through every section.

Plate 5.  
Fig. 15.

Let AB be the reservoir, BC the bottom of the river, and ZX, QR, sections of the river. Because the river flows uniformly, the same quantity of water which passes through ZX in a given time must pass through QR in the same time: otherwise the quantity of water in the space ZQXR, must in that time be increased or diminished, and consequently the depth of the water in that space altered; contrary to the supposition.

## P R O P. XIX.

The breadth of the channel being given, the water in rivers is accelerated in the same manner with any body moving down an inclined plane.

For each drop of the water moves down upon the inclined plane of the bottom, or upon the inclined plane of the sheet of water, next below it, parallel to the bottom.

## P R O P. XX.

The breadth of the channel being given, the velocity of each drop of water in a river is the same that a body would acquire in falling from the level of the surface of the water in the reservoir, to the place of the drop.

Let AB be the depth of the reservoir, AP the level of its surface, and BC the bottom of the channel. Any drop at E, after it comes out of the reservoir at K (by Prop. XIX.) rolls down the inclined plane KE parallel to the bottom. And this drop, when it comes out of the reservoir AB at K (by Prop. XIII.) has the same velocity which a heavy body would acquire in falling from A to K: and, in rolling down the inclined plane KE, it acquires (by Book II. Prop. XXXIV.) the same velocity which any heavy body would acquire in falling down GE the perpendicular height of the plane. At E the drop will therefore have acquired a velocity equal to that which a body would acquire by falling through AK and GE, that is, through MGE, the perpendicular drawn from the level of the reservoir to the place of the drop.

COR. I. Hence the breadth of the channel being given, the velocity of each drop of water in a river is as the square root of its distance from the level of the surface of the



the reservoir. For, if E and R be two drops in different parts of the river, and AP the level, the velocity of the drop E is the same that a body would acquire by falling down ME, and that of R the same which a body would acquire by falling down HR. Therefore (by Book II. Prop. XXVI. Cor. 1.) the velocity of the drop E is to the velocity of R, as the square root of ME to the square root of HR.

Plate 5.  
Fig. 15.

COR. 2. Hence the breadth of the channel being given, the water at the bottom of a river will run faster than the water at the surface.

## P R O P. XXI.

The breadth of the channel being given, the depth of the river continually decreases as it runs.

The same quantity of water (by Prop. XVIII.) passes through each of the sections ZX, QR, in the same time. But (by Prop. XX. Cor. 2.) the water runs faster at the lower section QR, than at the upper ZX. Therefore the area of the section QR must be as much less than the area of the section ZX, as the velocity at QR is greater than the velocity at ZX. But the breadth of the sections are by supposition equal; therefore their areas are (El. VI. 1.) as their heights. Consequently, the heights of the section QR, ZX, will be inversely as the velocities at those sections; that is, the depth of the water at QR will be as much less than the depth at ZX, as the velocity at QR is greater than the velocity at ZX.

## P R O P. XXII.

At a given distance from the reservoir, if the river flows uniformly, the velocity of the water will be inversely as the breadth of the channel.

Because the river flows uniformly, the depth at any given section ZX is always the same: and in any given time, the same quantity of water must flow through the different sections ZX, QR, as was shewn in Prop. XVIII. But a given quantity of water cannot flow in a given time through any section, unless as much as the area is increased, so much the velocity is diminished, and the reverse; that is, the velocity must be inversely as the area of the section, or the depth being given, as its breadth.

## P R O P. XXIII.

The depth of a river being given, the pressure upon any part of the bank will be the same, whatever is the breadth of the river.



The preffure upon any given part in the bank (by Prop. I. and III.) will be as the distance of that part from the surface; which remains the same whilst the depth is the same, whatever be the breadth of the river: therefore the preffure will remain the same.

## P R O P. XXIV.

If the breadth of a river be given, the preffure on any part of the bank will be as the depth of the river.

For the preffure on any part of the bank is (by Prop. I. and III.) as the depth of that part below the surface, which depth will increase with the depth of the river.

## P R O P. XXV.

The preffure against any given surface in the bank of a river, if that surface reaches from the bottom to the top of the stream, is equal to the weight of a column of water whose base is the surface, and whose height is half the depth of the stream.

Plate 5.  
Fig. 15.

Let  $ZQXR$  be a given surface in the bank reaching from the bottom  $BC$  of the river to its top  $AD$ . The preffure upon this is (from what was shewn in Prop. IX.) half the preffure on an equal surface at the bottom  $XR$ ; which preffure (by Prop. I. and III.) is equal to the weight of a column of water whose base is the surface  $ZQ$ , and whose height is the depth of the stream. Therefore the preffure against the surface  $ZQXR$  is equal to the weight of a column whose base is the surface  $ZQ$ , and its height half the depth of the stream.

## P R O P. XXVI.

When a stream which moves with the same velocity in every part strikes perpendicularly upon any obstacle, the force with which it strikes is equal to the weight of a column of the same fluid, whose base is the obstacle, and whose height is the space through which a body must fall to acquire the velocity of the stream.

Plate 5.  
Fig. 11.

Let a stream of water flow horizontally out of the orifice  $e$ . If this stream were to strike upon an obstacle of the same breadth every way as the orifice or stream, placed perpendicular to the horizon, the stream must strike upon the obstacle with its whole force. But this force is equal to the weight of a column of water whose base is  $e$ , and height  $Ae$ .

And



And (by Prop. XIII.)  $Ae$  is the height from which a body must fall to acquire the velocity with which the stream spouts from  $e$ . Therefore the force with which this stream would strike such an obstacle is equal to the weight of a column of water whose base is  $e$ , and height that from which a body must fall to acquire the velocity of the stream. And because no part of the stream, however broad, can strike the obstacle except so much as is contained within a section equal to the surface of the obstacle, no other part of the stream is to be considered in estimating this force. It is also manifest, that if the stream flow horizontally with the same velocity, in any other manner than through an orifice, as in the current of a stream, it will strike an obstacle with the same force.

## P R O P. XXVII.

When the obstacle is given, the force with which a stream strikes upon it, will be as the square of the velocity with which the stream moves.

If a stream strikes upon a given obstacle, the force will (by Prop. XXVI.) be equal to the weight of a column of water whose base is the obstacle, and whose height is equal to the space through which a body must fall to acquire the velocity of the stream. Since then the base is given, the weight will be as the height of such a column. But the spaces through which bodies fall to acquire different velocities are (by Book II. Prop. XXVI.) as the squares of those velocities. Therefore the height of this column, and its weight, and consequently the force of the stream, which is equal to this weight, will be as the square of the velocity with which the stream moves.



## C H A P. III.

*Of the RESISTANCE of Fluids.*

## P R O P. XXVIII.

If a spherical body is moving in a given fluid, the resistance which arises from the re-action of the particles of the fluid is as the square of the velocity with which the body moves.

A spherical body moving in a given fluid, the number of particles which it will meet with in a given time will be as its velocity; for the space through which it will pass will be as its velocity, and the number of particles it will meet with will be as the space through which it passes. But the re-action of the particles of the fluid, and consequently the resistance, is as the number of particles or quantity of matter by which the resistance is made. Again, if a given quantity of matter is to be moved, the moving force is (by Book II. Prop. IX.) as the velocity communicated; and the resistance of that given quantity of matter is as the moving force. Therefore the resistance arising from re-action in a given number of particles of fluid is as the respective velocities with which they are moved; that is, as the velocities with which the bodies which pass through the fluid move. The resistance of the fluid being then as the velocity on a double account, first, because the number of particles moved are as the velocity of the moving body, and secondly, because the resistance of a given number of particles is as the velocity of the moving body; the resistance will be in the duplicate ratio, or as the square of this velocity.

## P R O P. XXIX.

When a spherical body moves with a given velocity in any fluid, the resistance of the fluid arising from its re-action, will be as the squares of the diameter of the spherical body.

A spherical body, in moving through a fluid, displaces a cylindrical column of that fluid, the height of which is the space which the sphere describes, and its base a great circle of the spherical body. Because the velocity is given, the space described in a given time, that is, the length of the column is given: whence, the quantity of fluid in the column,



column, that is, the column will be as its base, a great circle of the sphere. And the resistance which the column of fluid makes by re-action to the motion of the sphere will be as its quantity of matter: it will therefore be as the base of the column, or as the great circle of the sphere, or (El. XII. 2.) as the square of its diameter.

## P R O P. XXX.

If two unequal homogeneous spheres are moving in the same fluid with equal velocities, the greater sphere will be less resisted in proportion to its weight, than the lesser sphere.

The weights of spheres, or their solid contents, are (El. XII. 18.) as the cubes of their diameters: but their resistances (Prop. XXIX.) are as the squares of their diameters: and the cubes of any numbers have a greater ratio to each other than their squares. Therefore the ratio of the weights of spherical bodies is greater than that of their resistances in a given fluid: that is, the weight of the greater sphere exceeds the weight of the lesser; more than the resistance of a given fluid against the former exceeds the resistance against the latter, provided the spheres are moving with equal velocities.

SCHOL. Hence the resistance of the air may be able to support small particles of fluid, but unable to support them when they are collected into larger drops.

## P R O P. XXXI.

The resistance of a fluid arising from its re-action, is as the side of the body perpendicularly opposed to it.

The resistance is as the column, or quantity of fluid removed in a given time, which, as was shewn Prop. XXIX. is as the base of the column, that is, as the side of the body perpendicularly opposed to it.

## P R O P. XXXII.

When equal spheres move with the same velocity in different fluids, the resistances will be as the densities of the fluids.

The resistances arising from re-action are as the momenta communicated to the fluid in a given time, that is, since the spheres move with equal velocities, as the quantities of matter moved. But, because the spheres are equal, the bases of the columns to which they communicate motion, are equal; and because the spheres move with equal velocity,  
the



the length of the columns to which they communicate motion are equal. Hence the columns to which motion is communicated, having their bases and heights equal, are of equal magnitude: and consequently, their quantities of matter are as their densities. But it has been shewn that their momenta and resistances are as their quantities of matter: therefore their resistances are as their densities.

SCHOL. Hence drops of water may be sustained in the lower parts of the atmosphere, which cannot be sustained in the higher.

P R O P. XXXIII.

The retardation of bodies in a resisting fluid, where the weights of the bodies are given, is as the resistance of the fluid.

The more a body is resisted by any fluid in which it moves, the greater portion of its momentum is destroyed; but, because the weight of the body is given, its momentum is as its velocity: therefore the greater the resistance of the fluid, the greater portion of its velocity is destroyed, that is, the more it is retarded.

P R O P. XXXIV.

When the resistance is given, the retardation is inversely as the weights.

The same resistance will destroy an equal portion of momentum whatever is the weight of the moving body. But when the momentum is the same, the velocity is (by Book II. Prop. XII.) inversely as the quantity of matter. Therefore the velocity destroyed, on the retardation, will be inversely as the quantity of matter in the body in which the momentum is destroyed: and the weight is as the quantity of matter: therefore the retardation is inversely as the weight.

P R O P. XXXV.

The retardation of spherical bodies moving with equal velocities in the same fluid, is inversely as their diameters.

The resistance which spherical bodies meet with in a given fluid is (by Prop. XXIX.) as the square of their diameter. The retardation, when the weight is given, is (by Prop. XXXIII.) as the resistance: and when the resistance is given, the retardation (by Prop.



Prop. XXXV.) is inverfely as the weight, that is, (El. XII. 18.) inverfely as the cubes of the diameter. Now, when unequal fpheres move with the fame velocity in the fame fluid, the retardations will be unequal, both becaufe the refiftances are unequal, and becaufe the weights are unequal. The retardations will therefore be directly as the fquares of the diameters, and inverfely as the cubes of the diameters: that is (compounding thefe ratios) inverfely as the diameters.

## P R O P. XXXVI.

When a body moves in an imperfect fluid which has tenacity, or the parts of which cohere, the refiftance of any given portion of the fluid from this caufe, is inverfely as the velocity of the body: the refiftance, when the velocity is given, is as the quantity of fluid through which the body paffes: and the refiftance is always as the time during which the body moves in the fluid.

Case 1. Suppose fuch an imperfect fluid, as foft clay, divided into thin plates; each plate having a certain portion of tenacity will continue to refift the body during the whole time in which it is paffing through it: the refiftance therefore will be the lefs, the fhorter time the body takes in paffing through it, that is the greater velocity the body moves with. And this is true concerning every plate which compofes the fluid. Therefore the refiftance arifing from tenacity in a given quantity of fluid, is inverfely as the velocity of the body which paffes through it.

Case 2. Again, the velocity of the body being given, the refiftance which the body meets with, from what has been faid, is alfo given, and will be as the number of plates or quantity of the fluid.

Case 3. Laftly, when a body moves for a given time, the refiftance (by the fecond cafe) is as the number of plates, that is, as the fpace through which it paffes in a given time, that is, (by Book II. Prop. VI.) as the velocity directly. And (by the firft cafe) the refiftance is, on account of the tenacity, inverfely as the velocity. Therefore as much as the refiftance is increafed on account of the velocity in one refpect, fo much it is diminifhed on account of the velocity in another; and confequently, whatever be the velocity of a body in fuch a fluid, the refiftance which it meets with in a given time will be the fame: whence this refiftance will be as the time in which the body moves in the fluid.



## C H A P. IV.

*Of the Specific Gravities of Bodies.*

DEF. V. The *density* of a body, is its quantity of matter when the bulk is given.

DEF. VI. The *specific gravity* of a body, is its weight compared with that of another body of the same magnitude.

COR. 1. The specific gravity of a body is as its density. For the specific gravity of a body is the weight of a given magnitude, and the weight of a body (by Book II. Prop. XXIV. Cor.) is as its quantity of matter; therefore the specific gravity of a body is as the quantity of matter contained in a given magnitude, that is, as its density.

COR. 2. The specific gravities of bodies are inversely as their magnitudes when their weights are equal. For by the last Cor. the specific gravities of bodies are as their densities, and their densities (from Def. I.) are inversely as their magnitudes when their weights are equal. Therefore the specific gravities are also inversely as their magnitudes when their weights are equal.

## P R O P. XXXVII.

A fluid specifically lighter than another fluid will float upon its surface.

For (by Book II. Prop. XXIV.) the lighter fluid will be less powerfully acted upon by the force of gravitation than the heavier; whence, the heavier will take the lower place.

EXP. 1. Let a small and open vessel of wine be placed within a large vessel of water; the wine will ascend.

2. Let mercury, water, wine, oil, spirits of wine, be put into a phial in the order of their specific gravities; they will remain separate.

## P R O P. XXXVIII.

The heights to which fluids, which press freely upon each other, will rise, are inversely as their specific gravities.

For



Since (by Prop. VI.) the opposite parts of an homogeneous mass of fluid, in a curved tube or channel, press equally against each other when they rise to the same height; in order to preserve the pressure equal when the fluids on each side are different, that which has the least specific gravity, must proportionally rise above the level to preserve the balance; and the reverse.

Exp. Into the longer arm of a recurved tube, of equal bore throughout, and open at each end, pour such a quantity of mercury, that it shall rise in each arm about half an inch; then pour water into the longer arm till the mercury is raised one inch above its former height; the specific gravities of these fluids will be inversely as the heights to which they rise.

## P R O P. XXXIX.

The force with which a body lighter than any fluid endeavours to ascend in that fluid, is as the excess of the specific gravity of the fluid above the solid.

Since ABCD, the fluid in a vessel, will be at rest (Prop. III.) when every part of an imaginary plane SQ, under the surface of the floating body *ptei*, sustains an equal pressure; if the solid body be of equal specific gravity with the fluid, that is, weighs as much as a quantity of the fluid equal to it in bulk, and whose place it takes up, this imaginary plane being equally pressed by the solid, as if the same space were filled with fluid, the fluid will be at rest, and the solid will neither ascend nor descend. Consequently, if the body be specifically heavier than the fluid, that part of the plane which is directly under the solid being so much more pressed than the other equal parts of the same plane as the solid body is specifically heavier than the fluid, the body must descend with a force equal to that excess; and, on the contrary, if the body be specifically lighter than the fluid, that part of the plane which is directly under the solid being so much less pressed than the other equal parts of the same plane, as the body is specifically lighter than the fluid, it must be buoyed up with a force equivalent to the difference of their specific gravities.

Plate 5.  
Fig. 19.

Exp. From a hollow cube of tin, let a cord fastened to one of its sides pass over a pulley fixed upon a perpendicular stem at the bottom of a vessel, and be connected with one arm of a balance: water being poured into the vessel, the cube will float on the surface; then add weights to the other arm of the balance, till the cube is entirely immersed: these weights will measure the force with which the cube endeavours to ascend. If the same be done with any other cube of the same bulk, and the specific gravities of the cubes be estimated, their forces will be found to be as the excesses of the specific gravity of the fluid above the specific gravities of the solids.

P 2

P R O P.



## P R O P. XL.

Any fluid presses equally against the opposite sides of a solid body immersed in it.

Plate 1.  
Fig. 18.

The opposite sides of the solid are at the same depth; and fluids at the same depth press equally. Thus the opposite sides RM, SN, of any body immersed in a vessel of water ABCD, are pressed equally by the surrounding fluid.

COR. No motion of the solid will be produced by these opposite lateral pressures.

## P R O P. XLI.

A body immersed in a fluid is pressed more upwards than it is downwards, and the difference of these two pressures is equal to the weight of as much of the fluid as would fill the space which the body fills.

Plate 5.  
Fig. 18.

The body MRNS being immersed in the vessel of water ABCD, its lower part MN must be pressed upwards just as much as the water itself at the same depth MNT would be if no solid were immersed. Now the water at any depth (by Prop. III.) is pressed as much upwards as it is pressed downwards. And at the depth MNT, the portion of this stratum MN, would, if the solid were away, be pressed downwards by a force equal to the weight of the incumbent column of water EMNH. Therefore the force with which MN, that is, the lower part of the solid, is pressed upwards, is equal to the weight of as much water as would fill the whole space EHMN. But the solid body RSMN is pressed downwards by the weight of the column above it EHRS. Therefore the difference between the two pressures, is the difference of the weights of the two columns of water EHMN, and EHRS: that is, the upward pressure upon the solid body RSMN exceeds the downward pressure, by a force equal to the weight of as much water as would fill the space RSMN, taken up by the solid body. The case will be the same whatever is the figure of the body immersed.

## P R O P. XLII.

A body immersed in a fluid, if it is specifically heavier than the fluid, will sink.

Plate 5.  
Fig. 18.

If the body RSMN is specifically heavier than the fluid, it weighs more than a quantity of the fluid of the same bulk with it. Hence the column EHMN, consisting of the column of fluid EHRS and the solid body RSMN, is heavier than the same column would be if it consisted



consisted wholly of water. But the upward pressure against MN is (by Prop. III.) equal to the downward pressure of the column of water EHMN, and therefore only sufficient to support the weight of that column. It cannot then support the weight of the heavier column, consisting of a fluid and a solid, EHMN; and that part of this column which is specifically heavier than the fluid, that is, the solid, will sink, with a force equal to the difference of the weights of the column of fluid EHMN, and the mixed column EHRS, RSMN.

## P R O P. XLIII.

A body specifically lighter than the fluid in which it is immersed, will rise to the surface and swim.

If the solid RSMN be a body specifically lighter than water, the column EHMN will weigh less as it consists of the column of water EHRS and the solid RSMN, than if it consisted entirely of water. Consequently, the upward pressure upon MN, which is equal to the weight of the column of water EHMN, will be equal to more weight than that of the mixed column EHRS, RSMN: and therefore the lighter part of this column, that is, the solid body, will be carried upwards with a force equal to the difference of the weights of the column of fluid EHMN, and the mixed column EHRS, RSMN.

Plate 5.  
Fig. 18.

## P R O P. XLIV.

A body which has the same specific gravity with the fluid in which it is immersed, will remain suspended in any part of the fluid.

The body RMNS being of the same specific gravity with the fluid, the column EHMN presses downwards with the same force, whilst this body makes a part of it, as if the column consisted wholly of water, that is, with a force equal to the upward pressure against MN. Therefore the body RSMN, having its lower surface MN, and in like manner all its parts, pressed by equal forces in opposite directions, will remain at rest.

Plate 5.  
Fig. 18.

Exp. Let small glass images made hollow, and of specific gravity somewhat less than water, having a small orifice to receive water, swim in a large glass vessel nearly filled with water and covered over closely with a piece of bladder: by pressing the bladder with the hand; the air on the surface is compressed; this pressure is communicated to the air in the images, which consequently receive a larger portion of water, and become in specific gravity as heavy as the water, or heavier, and either float in the water, or sink.

P R O P.



## P R O P. XLV.

A body specifically heavier than the fluid in which it is immersed, may be supported in it by the upward pressure, if the pressure downward be taken away: and a body specifically lighter than the fluid in which it is immersed, will not rise in the fluid, if the upward pressure be taken away.

Plate 5.  
Fig. 18.

For, in the first case, the pressure which the solid RSMN sustains from the weight of the fluid being removed, the solid may press downwards with a force equal to, or less than, that of the column of fluid EHMN, that is, than that with which it is pressed upwards, according to the degree of depth in the fluid at which the solid is placed.

In the second case, as the upward pressure against MN is diminished, the downward pressure of the mixed column EHMN becomes equal to, or greater than, the upward pressure, and the solid will either float in the fluid, or sink.

Exp. For the first part of the proposition see Prop. III. Exp. 2 and 5. The second part may be thus confirmed. If a plane and smooth piece of hard wood, or of cork, be closely pressed down by the hand upon the plane and smooth bottom of any vessel, whilst mercury is pouring into the vessel; upon removing the pressure of the hand, the downward pressure of the mercury will prevent the wood from rising.

## P R O P. XLVI.

If a body floats on the surface of a fluid specifically heavier than itself, it will sink into the fluid till it has displaced a portion of fluid equal in weight to the solid.

Plate 5.  
Fig. 19.

Let *ptei* be a body, floating on a liquor specifically heavier than itself, it will sink into it, till the immersed part, *rnei*, takes up the place of so much fluid as is equal to it in weight. For, in that case, *ei*, that part of the surface of the stratum upon which the body rests, is pressed with the same degree of force, as it would be, was the space *rnei* full of the fluid; that is, all the parts of that stratum are pressed alike, and therefore the body, after having sunk so far into the fluid, is in equilibrio with it, and will remain at rest.

Exp. I. Place a cube of wood on a small jar, exactly filled with water; a part of its bulk will be immersed, and will displace a quantity of the water: take the cube out of the water, and put it into a scale, with which an empty vessel in the other scale stands balanced; then



then pour water into that vessel till the equilibrium is restored; that portion of water will fill up the jar in which the cube was placed.

2. Let a glass jar, with a weight sufficient to make it sink in water to about two-thirds of its length, be placed first in a large vessel of water, and afterwards in one which is very little wider than the jar, and which has in it a small quantity of water; the jar will sink to the same depth in both vessels, that is, till so much of the vessel is under water as is equal in bulk to a quantity of the fluid whose weight is equal to that of the whole vessel.

COR. Hence arises a rule for estimating the specific gravities of fluids or solids. For, since (by this Prop.) the weights of the water displaced and of the solid are equal, their specific gravities are inversely as their magnitudes; that is, the magnitude of the water displaced is to that of the solid, as the specific gravity of the solid is to the specific gravity of the fluid: or (since the part immersed is equal in magnitude to the fluid displaced) the part immersed is to the whole, as the specific gravity of the solid to the specific gravity of the fluid. Consequently, the greater portion of any given solid is immersed in any fluid, the less is the specific gravity of the fluid: and with respect to solids, inverting the proposition, as the whole is to the part immersed, so is the specific gravity of the fluid to that of the solid: whence, the greater portion of any solid is immersed in a given fluid, the greater is its specific gravity.

EXP. 1. Let an hydrometer (or bulb, with a graduated stem, specifically lighter than any of the fluids to be measured) be immersed in different fluids: the deeper it sinks, the less is the specific gravity of the fluid.

2. Let two cylinders of wood, equal in length and diameter, one of oak, and the other of fir, be immersed in narrow tall jars of water; their different specific gravities will be seen from the different depths to which they sink.

#### P R O P. XLVII.

A solid weighs less when immersed in a fluid than in open air, by the weight of a quantity of the fluid equal in bulk to the solid.

If the body immersed were of the same specific gravity with the fluid (by Prop. XLIV.) it would be supported in the fluid by the upward pressure. The fluid therefore sustains so much of the gravity of the body, or takes away so much of its weight, as is equal to the weight of that quantity of fluid which would fill the place taken up by the body.

Or thus; A body endeavours to descend by its whole weight; but (as was shewn Prop. XLI.) when it is immersed in a fluid, it is supported by a force equal to the weight of an equal bulk of that fluid. And since these two forces act in contrary directions, the weight which the body retains in the fluid will be the difference between them; that



that is, it weighs as much less in the fluid as in the air, as the weight of a quantity of the fluid equal in bulk to the solid.

EXP. Having provided a solid cylinder of lead which exactly fills a hollow cylinder of brass, place in one scale the hollow cylinder; under the same scale suspend by a string the solid cylinder, and balance the whole by weights; then immerse the solid cylinder in water, and the equilibrium will be restored by filling up the hollow cylinder.

COR. 1. Hence the specific gravities of different fluids may be compared, by observing how much the same solid (specifically heavier than the fluids) loses of its weight in each fluid; that fluid having the greatest specific gravity in which it loses most of its weight.

EXP. Let a cubic inch of wood, made sufficiently heavy to sink in water, be immersed successively in different fluids; it will displace a cubic inch of the fluid in which it is immersed; and since the cube (by Prop. XLVIII.) weighs less in the fluid, by the weight of a quantity of the fluid equal in bulk to the cube, its loss of weight will be the weight of a cubic inch of the fluid.

COR. 2. The weights which bodies lose in any fluid are proportional to their bulks.

EXP. 1. Two balls of equal bulk, one of ivory, the other of lead, will lose equal weight in water.

2. A piece of copper and a piece of gold being of equal weight in air, the gold outweighs the copper in water.

COR. 3. If it be known what a cubic inch of any body loses in water, the solid contents of any irregular mass of the same kind may be known, by observing how much more or less it loses, than a cubic inch would lose.

EXP. Weigh a cubic inch and any irregular piece of wood of the same kind, and observe the difference of their weights.

COR. 4. The weight of a solid body of the same specific gravity with the fluid, or of a portion of the fluid itself, suspended in the fluid, is not perceived, because this weight is supported, and not because the gravity of the body is lost or destroyed.



B O O K III. P A R T II.

O F P N E U M A T I C S.

C H A P. I.

*Of the WEIGHT and PRESSURE of the AIR.*

DEF. The *Air*, or *Atmosphere*, is that fluid which encompasses the earth.

P R O P. XLVIII.

The air has weight.

This appears from experiment.

EXP. I. The air being exhausted, by an air-pump, from a glass receiver, the vessel will be held fast by the pressure of the external air.

2. If a small receiver be placed under a larger, and both be exhausted, the larger will be held fast, whilst the smaller will be easily moved.

3. If the hand be placed upon a small open vessel, in such manner as to close its upper orifice, it will be held down with great force.

4. The upper orifice of an open receiver being closely covered with a piece of bladder, upon exhausting the receiver, the bladder will burst.

5. In the same situation a thin plate of glass will be broken.

6. Let a glass bulb, having a long neck, be put, with the neck downwards, into a vessel of water; put the whole under a receiver, and exhaust the air: on re-admitting the air, its pressure upon the surface of the water in the vessel will cause the water to rise in the bulb.

7. Pour mercury into a wooden cup, closely placed upon the upper orifice of an open receiver: when the air beneath is exhausted, the pressure of the external air will force the mercury through the wood, and it will descend in a shower.

8. On a transferrer let the air be exhausted from a long receiver; then let water be admitted through a pipe, by means of a cock; the water will rise in a *jet d'eau*.

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9. Con-



9. Condense the air within a globular vessel, having a long neck, by blowing through the neck, the pressure of the air within the vessel will force out the water.

10. The glass bulb and vessel (as in Exp. 6.) being placed within a condensing receiver; on increasing the quantity of air in the receiver, the fluid will rise into the bulb.

11. The quicksilver in the gauge of the condenser will, by the increasing pressure of the air, be forced upwards in the tube.

12. Any quantity of air may be transferred from one vessel to another.

13. A thin glass vessel, of a cubical form and closed up, may be broken in a condenser.

14. Fill a glass tube, about 3 feet long and closed at one end, with mercury; then insert the open end in a vessel of mercury; the mercury will remain suspended in the tube, by the pressure of the external air upon the surface of the mercury in the vessel: when this pressure is removed, by placing the tube and vessel under a receiver and exhausting the air, the mercury will sink in the tube, and on re-admitting the air, will rise.

15. If the same immersed tube be suspended from the beam of a balance, the weight necessary to counterpoise it, exclusive of the weight of the tube, is equal to that of the mercury sustained in the barometer by the pressure of the atmosphere: for, the weight of the column of air incumbent upon the tube not being counterbalanced by the contrary pressure from below, which is employed in bearing up the mercury within the tube, must press upon the beam.

16. Let a barometer tube, instead of being hermetically sealed at the top, be closely covered with a piece of bladder; the mercury will rise to the same height as in a common barometer; and on piercing the bladder with a needle, to admit the air, it will fall.

SCHOL. Hence the pressure of the atmosphere on or near the surface of the earth is known: the weight of any column of air being equal to the weight of the column of mercury, of the same diameter, supported in the barometer. And, since the height of this column varies with the weight of the atmosphere, the varieties in the weight of the atmosphere are known by the barometer.

18. Let the air be exhausted from a glass vessel, and by means of a cock let the vessel be kept exhausted; weigh the vessel whilst it is exhausted, and when the air is re-admitted; the difference is the weight of so much air as the vessel contains.

#### P R O P. XLIX.

The air presses equally in all directions.

Exp. 1. Let a bladder, filled with air, be placed within a condensing receiver, the condensed air will make the bladder flaccid.

2. In a tall phial let an orifice be made about 3 inches above the bottom; stop this orifice; through a cork in the neck of the phial insert a long tube open at each end; and let its lower end be below the orifice in the side of the phial. The mouth of the phial being closed up about the tube, pour water into the tube till it is full. Upon opening the



the orifice, the water will be discharged till its surface in the tube is level with the orifice; after which it will cease to flow, because the external lateral pressure of the air balances the perpendicular pressure upon the water in the tube.

3. If a glass vessel be filled with water, and covered with a loose piece of paper, on inverting the glass, the water will be kept from falling by the upward pressure of the air.

4. If a vessel be perforated in small holes at the bottom, but closed at the top, the upward pressure of the air will keep the water within the vessel; as will appear by successively stopping and unstopping a small hole in the top of the vessel.

5. Two brass hemispherical cups put close together, when the air between them is exhausted, will be pressed together with considerable force.

6. A syringe being fastened to a plate of lead, and the piston of the syringe being drawn upwards with one hand, whilst the lead is held in the other, the air, by its upward pressure, will drive back the syringe upon the piston: whereas, if the loaded syringe be hung in a receiver, and the air be exhausted, the syringe and lead will descend; but upon re-admitting the air, they will again be driven upwards.

P R O P. L.

The pressure of the atmosphere varies at different altitudes.

EXP. Put a glass tube, open at both ends, through a cork into a large phial containing a small quantity of coloured water; let the lower end of the tube be in the water; and let the cork and tube be closely cemented to the neck of the bottle. Then, blow through the tube, till the quantity of air within the phial is so increased, that the water will rise above the neck of the phial. Let this phial be placed in a vessel of sand, to keep the air within of the same temperature; the water will stand at different heights in the tube, according to the elevation of the place where it is placed: from whence it appears, that the pressure of the atmosphere varies at different altitudes.

COR. Hence the proportion of the specific gravity of air to that of water may be determined. If the difference in height of the two places where the above experiment is made be 54 feet, and that difference cause a difference of  $\frac{3}{4}$  of an inch in the height of the water; it follows, that a column of water of  $\frac{3}{4}$  of an inch or  $\frac{1}{16}$  of a foot, is equi-ponderant to a column of air of 54 feet having the same base: therefore the gravity of air to that of water, is as 54 to  $\frac{1}{16}$ , or 864 to 1. In ascending the mountain of Snowden in Wales, which is 3720 feet perpendicular height, it was found that the barometer sunk 3 inches and  $\frac{8}{16}$ .



## P R O P. LI.

The force with which the wind strikes upon the sail of a ship, the velocity of the air and the dimension of the sail being given, will be as the square of the cosine of the angle of incidence.

Plate 5.  
Fig. 16.

Let AD represent the sail of a ship, with its edge towards the eye; and let a circle be drawn upon the center K; whence K will be the middle of the sail, and AD its length. If the wind blows perpendicularly against the sail, all the air included within the space FADG will strike upon it. But if the sail is inclined in the position BE, all the air, which strikes upon it, is included within the space HBEI.

If it were possible that the sail should be struck with the same quantity of air in the perpendicular position AD as in the oblique position BE, yet the quantity of the oblique stroke would be to the quantity of the direct stroke (by Book II. Prop. XIX.) as the cosine of incidence to the radius, that is, since (supposing LK drawn parallel to BH, the direction of the wind, and BC perpendicular to KL) BKL is the complement of the angle of incidence LKE, and BL its cosine, as BL to AK.

Again, if it were possible that the oblique stroke of the wind upon the sail BE should be equal to the direct stroke upon AD; yet, the column of air which strikes upon the sail directly, having AD for its base, and the column which strikes obliquely, having BC for its base, the quantity of air which strikes obliquely, is to that which strikes directly, as BC to AD, that is, as BL to AK; but the velocities in either case are supposed to be the same: therefore the momenta, or forces with which the sails are struck will be as the quantities of matter, that is, as BL, the cosine of incidence to AK the radius.

Thus, the force with which the wind strikes the sail BE obliquely, is to the force with which it strikes an equal sail AD directly, as BL to AK on two accounts; first, because an oblique stroke is to a direct stroke in this ratio; and secondly, because the quantity of air which strikes the oblique sail is to that which strikes the direct one in the same ratio. Consequently, upon both accounts together, the oblique force is to the direct one as  $BL \times BL$  to  $AK \times AK$ , or as the square of BL the cosine of incidence to the square of AK the radius. But, the length of the sail, or AD being given, AK the radius is a given quantity. Therefore the force of the wind in different obliquities of the sail, will be as BL the square of the cosine of incidence.



## C H A P. II.

*Of the* ELASTICITY *of the* AIR.

## P R O P. LII.

The air is an elastic fluid, or capable of compression and expansion.

EXP. I. A blown bladder, pressed with the hand, will return into the form which it had before the pressure.

2. A flaccid bladder, put under a receiver, when the external air is exhausted, becomes extended by the elasticity of the internal air.

3. A bladder suspended within the receiver, with a small weight hanging from it which touches the bottom, when the external air is exhausted, by the expansion of the internal air, will raise the weight.

4. The bladder being put into a box, and a weight laid upon the lid, the weight, on exhausting the air, will be lifted up.

5. If a tube, closed at one end, be inserted at its open end in a vessel of water, the fluid in the tube will not rise to the level of the water in the vessel, being resisted by the elastic force of the air within the tube. On this principle the diving bell is formed.

6. If a bladder be inclosed in a glass vessel so closely that the air in the vessel without the bladder cannot escape, but the air within the bladder communicates with the external air through the neck of the vessel; the external air being exhausted, the bladder will be closely pressed by the air in the vessel; and when the air is re-admitted, the bladder will be distended.

7. A shrivelled apple, under an exhausted receiver, will have its coat distended by the internal air.

8. In the same situation, the air contained in a fresh egg will expel its contents from an orifice made in its smaller end.

9. On green vegetables, and other substances, placed in a vessel of water under a receiver, whilst the air is exhausting, bubbles will be raised by the expansion of the internal air.

10. Beer, a little warmed, will, from the same cause, whilst the internal air is exhausting, have the appearance of boiling.

11. Let a cylindrical piece of wood (made just specifically heavier than water by fastening to it a small plate of lead) be placed in a vessel of water under a receiver: upon exhausting



exhausting the air the wood will swim; some particles of air escaping from the wood, and hereby diminishing its specific gravity.

12. The bulb inserted in a vessel of water (as in Prop. XLVIII. Exp. 6.) being nearly filled with water by exhausting the air, on its re-admission, the air within the bulb, by its elasticity, will expel the water from the bulb.

13. Place a double transferrer upon the air-pump, with two receivers; exhaust one receiver; then open the pipe between the two receivers; and the air in the unexhausted receiver will, by its elasticity, be in part driven into the exhausted receiver; and both receivers will have equal portions of air; but this air will be rarer in both than the external air; whence both the receivers will be held fast by the external pressure.

### P R O P. LIII.

The elastic spring of the air is equivalent to the force which compresses it.

If the spring with which the air endeavours to expand itself when it is compressed were less than the compressing force, it would yield still farther to that force; if it were greater, it would not have yielded so far. Therefore, when any force has compressed the air so that it remains at rest, the spring of the air arising from its elasticity can neither be greater nor less than this force that is, must be equal to it.

EXP. Let the air be exhausted from an open tube, whose lower part is inserted in a vessel containing a small quantity of mercury, and let the air within the vessel be prevented from escaping; this air, by its elasticity, will force the mercury up the tube nearly to the height to which it would be raised by the pressure of the atmosphere.

### P R O P. LIV.

The space which any given quantity of air fills is inversely, and its density directly, as the force which compresses it.

EXP. Let there be a bent tube of the form  $nkg$ , open at  $n$  and closed at  $g$ . Let a small portion of mercury be at the bottom  $ki$ . Then  $gi$  is filled with air compressed by the weight of the atmosphere, equivalent to the weight of a column of mercury about  $29\frac{1}{2}$  inches in height. If more mercury be poured into the orifice  $n$ , the weight of this mercury is an additional compressing force acting upon the air  $ig$ . Since (by Prop. V.) the columns of equal heights  $lk$ ,  $hi$ , balance each other, the air in the space  $gi$ , is pressed both by the weight of the atmosphere and the column  $ml$ . If therefore  $ml$  be  $29\frac{1}{2}$  inches, the air in  $gi$  is pressed with double the weight of the atmosphere,  
or



or with two atmospheres; and it will be found, that it will be compressed into the space  $gb$ , half the space which the same quantity of air took up when it was pressed only with the weight of the atmosphere: therefore the space is inversely as the compressing force. And its density (Def. V.) is inversely as its bulk, or the space filled by it. Since therefore, both the compressing force and the density of the air are inversely as the space, the density must be directly as the compressing force.

P R O P. LV.

The air consists of particles, which repel each other with forces which are inversely as the distances between their centers.

An elastic fluid equally compressed in all directions must have all its particles at equal distances from each other: for if the distances are unequal, where it is the least, the repelling force will be greatest, and the particles will move towards the side where there is less repulsion, till the forces become equal, that is, till the particles are equally distant, or the fluid becomes every where of the same density. Suppose, then, two equal cubes of air, A and B; it is manifest, from the nature of the cube, that the number of particles in the whole mass A is equal to the cube of the number of particles in the line  $de$ ; and, in like manner, that the number of particles in the mass B is equal to the cube of the number of particles in the line  $hi$ . And the density of these two equal cubes of air A and B will be as the number of particles contained in them. Therefore the density of the cube A is to the density of the cube B, as the cube of the number of particles in the line  $de$  to the cube of the number of particles in the line  $hi$ . But, since these lines  $de$ ,  $hi$ , are of a given length, the number of particles in each, will be greater as the distances between their centers is less, that is, will be inversely as those distances. Whence, the cube of the number of particles in  $de$ ,  $hi$ , will be inversely as the cube of the distance between their centers. And it has been shewn, that the density of the mass A is to the density of the mass B, as the cube of the number of particles in  $de$  to the cube of the number of particles in  $hi$ . Therefore the density of A is to the density of B, inversely as the cube of the distance between the centers of the particles.

Plate 5.  
Fig. 17.

Also, in compressing any mass, A, every surface, as  $defg$ , is pressed closer to the surface next beyond it. And the repulsion of the surface  $defg$  against the surface next beyond it will be (all other circumstances being equal) as the number of repelling particles in that surface, that is, as the square of the number of particles in the line  $de$ . But the number of particles in the line  $de$  is inversely as the distance between their centers. Therefore the square of the number of particles in  $de$ , that is, the number of repelling particles in the surface  $defg$ , that is, the repulsion of this surface against the next beyond it, is inversely as the square of the distance between the particles. Again, where the number of particles in each surface is given, if it be supposed that the particles repel each other



other with a force which is inversely as the distance between their centers, since the surfaces are at the same distance from each other with the particles which compose them, the repulsion of the surfaces must be in the same ratio. Thus, the repulsion in the mass A is to that in the mass B, inversely as the distances of the particles, if only their approach to each other be considered. And it has been shewn that the repulsion is inversely as the square of these distances, if only the number of particles be considered. Therefore on both accounts taken together, the repulsion is inversely as the cube of the distance of the particles. And (by Prop. LIII.) the compression is as the repulsion: therefore the compression is inversely as the cube of the distance of the particles.

Now it was shewn above, that the density of A is to the density of B, inversely as the cube of the distance of the particles. Therefore, when a fluid consists of particles which repel each other with forces inversely as the distances between the centers of the particles, the density of the fluid will be as the compressing force. But it was shewn (Prop. LIV.) that the density of the air is as the compressing force. Therefore the air consists of particles which repel each other with forces which are inversely as the distances between their centers.

SCHOL. From the doctrine of the elasticity of the air, the phenomena of Sound may be explained.

When the parts of an elastic body are put into a tremulous motion, by percussion, or the like, as long as the tremors continue, so long is the air included in the pores of that body, and likewise that which presses upon its surface, affected with the like tremors and agitations. Now, the particles of air being so far compressed together by the weight of the incumbent atmosphere as their repulsive forces permit, it follows, that those which are immediately agitated by the reciprocal motions of the particles of the elastic body, will, in their approach towards those which lie next them, impel these also towards each other, and hereby cause them to be more condensed than they were by the weight of the incumbent atmosphere, and in their return will suffer them to expand themselves again: hence the like tremors and agitations will be propagated to them; and so on, till having arrived at a certain distance from the body, the vibrations cease, being gradually destroyed by a continual successive propagation of motion to fresh particles of air throughout their progress.

Thus it is that sound is communicated from a tremulous body to the organ of hearing. Each vibration of the particles of the sounding body is successively propagated to the particles of the air, till it reaches those which are contiguous to the *tympanum* of the ear, (a fine membrane distended across it) and these particles, in performing their vibrations, impinge upon the tympanum, which agitates the air included within it; which being put into a like tremulous motion, affects the auditory nerve, and thus excites in the mind the sensation or idea of what we call *sound*.

Now,



Now, since the repulsive force of each particle of air is equally diffused around it every way, it follows, that when any one approaches a number of others, it not only repels those which lie before it in a right line, but the rest laterally, according to their respective situations; that is, it makes them recede every way from itself, as from a center. And this being true of every particle, the tremors will be propagated from the sounding body in all directions, as from a center: and further, if they are confined for some time from spreading themselves by passing through a tube, or the like, will, when they have passed through it, spread themselves from the end in every direction. In like manner, those which pass through an hole in an obstacle they meet with in their way, will afterwards spread themselves from thence, as if that was the place where they began; so that the sound will be heard in any situation whatever, that is not at too great a distance.

Since the repulsive force with which the particles of air act upon each other, is reciprocally as their distances (by Prop. LV.) it follows, that when any particle is removed out of its place by the tremors of a sounding body, or the vibrations of those which are contiguous to it, it will be driven back again by the repulsive force of those towards which it is impelled, with a velocity proportional to the distance from its proper place, because the velocity will be as the repelling force. The consequence of this is, that, let the distance be great or small, it will return to its place in the same time; for the time a body takes up in moving from place to place will always be the same, whilst the velocity it moves with is proportional to the distance between the places. The time therefore in which each vibration of the air is performed, depends on the degree of repulsion in its particles, and so long as that is not altered, will be the same at all distances from the tremulous body; consequently, as the motion of sound is owing to the successive propagation of the tremors of a sounding body through the air, and as that propagation depends on the time each tremor is performed in, it follows, that the velocity of sound varies as the elasticity of the air, but continues the same at all distances from the sounding body.

Moreover, since the undulatory motion of the air, which constitutes sound, is propagated in all directions from the sounding body; it will frequently happen, that the air, in performing its vibrations, will impinge against various objects, which will reflect it back, and so cause new vibrations the contrary way; now, if the objects are so situated, as to reflect a sufficient number of vibrations back to the same place, the sound will be there repeated, and is called an *echo*. And, the greater the distance of the objects is, the longer will be the time, before the repetition is heard. And when the sound in its progress meets with objects, at different distances, sufficient to produce an echo, the same sound will be repeated several times successively, according to the different distances of those objects from the sounding body; which makes what is called a *repeated echo*.

If the vibrations of the tremulous body are propagated through a long tube, they will be continually reverberated from the sides of the tube into its axis, and by that means

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prevented



prevented from spreading, till they get out of it; whereby they will be exceedingly increased, and the sound rendered much louder than it would otherwise be; as in the *Speaking Trumpet*.

The difference of *musical* tones depend on the different number of vibrations communicated to the air, in a given time, by the tremors of the sounding body; and the quicker the succession of the vibrations is, the acuter is the tone, and the reverse.

### P R O P. LVI.

The elasticity of air is increased by heat.

EXP. To the bottom of a hollow glass ball let an open bended tube be affixed. Let the lower part of the bended tube and part of the ball be filled with mercury: the external surface will be pressed by the weight of the atmosphere; and the internal surface will be equally pressed by the spring of the air inclosed within the vessel. If the ball be immersed in boiling water, the increased elasticity of the included air will raise the mercury in the small tube. The same may be shewn by immersing in boiling water a tube, closed at one end, into which a small quantity of mercury has been admitted, inclosing a portion of air within the tube.

SCHOL. I. The Wind is no other than the motion of the air upon the surface of the globe. The principal cause of the wind is, that the atmosphere is heated over one part of the earth more than over another. For, in this case, the warmer air being rarefied, becomes specifically lighter than the rest; it is therefore overpoised by it, and raised upwards, the upper parts of it diffusing themselves every way over the top of the atmosphere; while the neighbouring inferior air rushes in from all parts at the bottom; which it continues to do, till the equilibrium is restored. Upon this principle it is, that most of the winds may be accounted for.

Under the *Equator*, the wind is always observed to blow from the east point. For, supposing the sun to continue vertical over some one place, the air will be most rarefied there; and consequently, the neighbouring air will rush in from every quarter with equal force. But, as the sun is continually shifting to the westward, the part where the air is most rarefied, is carried the same way; and therefore the tendency of all the lower air, taken together, is greater that way, than any other. Thus the tendency of the air towards the west, becomes general, and its parts impelling one another, and continuing to move till the next return of the sun, so much of its motion, as was lost by his absence, is again restored, and therefore the easterly wind becomes *perpetual*.

On each side of the *Equator*, to about the thirtieth degree of latitude, the wind is found to vary from the east point, so as to become north-east on the northern side, and south-east on the southern. The reason of which is, that, as the *equatorial* parts are hotter than any other, both the northern and southern air ought to have a tendency that way; the northern current therefore, meeting in this passage with the eastern, produces a north-east wind



wind on that side; as the southern current, joining with the same, on the other side the *Equator*, forms a south-east wind there.

This is to be understood of open seas, and of such parts of them as are distant from the land; for near the shores, where the neighbouring air is much rarefied, by the reflection of the sun's heat from the land, it frequently happens otherwise; particularly on the *Guinea* coast, the wind always sets in upon the land, blowing westerly instead of easterly. This is because the deserts of *Africa* lying near the *Equator*, and being a very sandy soil, reflect a greater degree of heat into the air above them; which being thus rendered lighter than that which is over the sea, the wind continually rushes in upon the land to restore the equilibrium.

That part of the ocean, which is called the *Rains*, is attended with perpetual calms, the wind scarcely blowing sensibly either one way or other. For this tract being placed between the westerly wind blowing from the ocean towards the coast of *Guinea*, and the easterly wind blowing from the same coast to the westward thereof, the air stands in equilibrio between both, and its gravity is so much diminished thereby, that it is not able to support the vapour it contains, but lets it fall in continual rain, from whence this part of the ocean has its name.

There is a species of winds, observable in some places within the *Tropics*, called by the sailors *Monsoons*, or Trade Winds, which, during six months of the year, blow one way; and the remaining six the contrary. The occasion of them in general is this: when the sun approaches the northern *Tropic*, there are several countries, as *Arabia*, *Persia*, *India*, &c. which become hotter, and reflect more heat than the seas beyond the *Equator*, which the sun has left; the winds therefore, instead of blowing from thence to the parts under the *Equator*, blow the contrary way; and when the sun leaves those countries, and draws near the other *Tropic*, the winds turn about, and blow on the opposite point of the compass.

From the solution of the general trade winds, we may see the reason, why in the *Atlantic* ocean, a little on this side the thirtieth degree of north latitude, there is generally a west, or south-west wind. For, as the inferior air, within the limits of those winds, is constantly rushing towards the *Equator*, from the north-east point, or nearly so, the superior air moves the contrary way; and therefore, after it has reached these limits, and meets with air, that has little or no tendency to any one point more than to another, it will determine it to move in the same direction with itself.

In our own climate we frequently experience, in calm weather, gentle breezes blowing from the sea to the land, in the heat of the day; which phenomenon is very agreeable to the principle laid down above: for the inferior air over the land being rarefied by the beams of the sun, reflected from its surface, more than that which impends over the water, the latter is constantly moving on to the shore, in order to restore the equilibrium, when not disturbed by stronger winds from another quarter.

From what has been observed, nothing is more easy than to see, why the northern and southern parts of the world, beyond the limits of the trade winds, are subject to such



variety of winds. For the air, upon account of the leffer influence of the sun in those parts, being undetermined to move towards any fixed point, is continually shifting from place to place, in order to restore the equilibrium, wherever it is destroyed, by the heat of the sun, the rising of vapours or exhalations, the melting of snow upon the mountains, or other circumstances.

EXP. Fill a large dish with cold water; into the middle of this put a water-plate, filled with warm water. The first will represent the ocean; and the other an island, rarefying the air above it. Blow out a wax candle, and if the place be still, on applying it successively to every side of the dish, the smoke will be seen to move towards the plate. Again, if the ambient water be warmed, and the plate filled with cold water, let the smoking wick of the candle be held over the plate, and the contrary will happen.

SCHOL. 2. Heat expands all bodies, solid as well as fluid.

EXP. 1, 2. Water may be rarefied into steam, and will become exceedingly elastic, acting with great power, as in the eolipile, and in steam engines.

3. Metals expand by heat; the degrees of their expansion are measured by the pyrometer.

4. Mercury, expanding or contracting by an increase or decrease of heat in the air, is made the measure of heat in thermometers.

SCHOL. 3. It is found by experiment, that air is necessary to the existence of sound, of animal life, of fire, and of explosion.

EXP. 1. Let a bell ring under an exhausted receiver, and in a condenser.

2. Let a lighted candle be extinguished under a receiver.

3. Let gun-powder fall upon red hot iron placed within an exhausted receiver.

SCHOL. 4. The component parts of that heterogeneous mass of fluids which we call the atmosphere, have been of late examined with great success; but to relate the result of these investigations, rather belongs to the science of chemistry, than to that of natural philosophy.

## P R O P. LVII.

To explain the nature and use of fundry Hydraulic and Pneumatic Instruments.

### I. The SYPHON.

Plate 5.  
Fig. 23.

Let DEC be a bended tube, having one leg longer than the other. This instrument, used for drawing off liquors, is called the syphon. If the shorter leg of the tube be inserted in a vessel of fluid, and if by sucking with the mouth a vacuum be produced in  
the



the tube, or if the tube be filled with the fluid before it is used, the fluid will run off from the vessel. The cause of which may be thus explained: the orifice C, of the longer leg, is exposed to the pressure of the atmosphere: also, since the fluid within the shorter leg is supported by the surrounding fluid in the vessel, the pressure upon the orifice D is that of the atmosphere. The two equal orifices are then acted upon by equal pressures; the difference of the lengths of the columns of atmosphere being too small to cause any perceptible difference in their pressure. But these equal pressures are counteracted by the pressures of two unequal columns of fluid ED, EC. If therefore the pressures of the columns of atmosphere be more than sufficient to balance those of the columns of fluid, that which acts with the lesser force, that is, the lesser column DE is more pressed against the column CE, than the column CE is pressed against DE at the vertex E. Consequently, the column EC must yield to the greater pressure, and flow off through the orifice C.

EXP. I. Draw off water by a syphon.

2. Whilst mercury is passing off from a vessel by a syphon, let the air be exhausted from the vessel, and the fluid will cease to run.

3. Intermitting fountains are natural syphons.

## II. The SYRINGE.

Let a hollow cylindrical tube have a small orifice at one end: at the other end insert a solid cylinder so exactly fitted to the tube, that no air can pass along its sides, and fix a handle to the solid cylinder. If that end of this instrument which has the smaller orifice be inserted in water, and the solid cylinder, or piston, be drawn back, a vacuum will be produced within the syringe; and the pressure of the atmosphere on the surface of the water, meeting with no opposite pressure, will force the water into the tube, from whence it may be forcibly expelled, by pushing down the piston.

## III. The COMMON PUMP.

In this useful instrument, a handle, acting upon a pin as a lever of the first kind, draws up a piston AD, fitted to the shaft or barrel of the pump, as described in the syringe. This piston has an hole, over which is a valve of leather, loaded with lead, opening upwards. Towards the lower part of the shaft is inserted a plug C, which also has in it a hole, and a valve which opens upwards. When the piston, or sucker, is drawn up from the plug, a vacuum is produced in the shaft between D and C, into which the air contained in the lower part of the pipe expands itself. By repeated strokes the air escapes through the upper valve, and the vacuum becomes so perfect, that the external air, pressing without counteraction, upon the surface of the water, in the well or reservoir in which the shaft is supposed to be inserted, forces the water through the valves at C and D, into the space AD; from whence it is prevented from returning downward, by the valves, which

Plate 5.  
Fig. 21.



which are closely pressed down by the incumbent fluid. If therefore the handle be repeatedly lifted up, the column of water will increase upon every stroke, till it rises to the level of the spout, and is discharged. But if the height be more than 34 feet, the water cannot be raised: for such a column is equal to the weight of a column of the atmosphere of the same diameter.

#### IV. The FORCING PUMP.

Plate 5.  
Fig. 22.

In this pump, the piston is one entire cylinder, as in the syringe. The water is raised into the pipe between A and D, as in the common pump: from hence it is forced, by the downward pressure of the piston, or forcer, through a tube inserted in the side of the main shaft. In this side-tube a valve is inserted at E to prevent the water from returning, and when a sufficient quantity is raised, it is discharged by the spout.

The common engine for *extinguishing fires* consists of two such forcing pumps, which convey the water into a reservoir made air-tight, into which a pipe is inserted. As this reservoir fills with water, the air within it is proportionally condensed, and therefore forces the water up a cylinder from which it is conveyed, at pleasure, by leathern pipes.

#### V. The CONDENSER.

This instrument, which is used to force air into any vessel, is a syringe, having a solid piston, and a valve in the lower part of its barrel which opens downwards. By thrusting down the piston, the air is forced through the valve, which is afterwards held close by the elasticity of the condensed air. When the piston is lifted up, a vacuum is produced, till it is raised above a small hole in the barrel, when the air rushes in, and is again discharged through the valve.

ARTIFICIAL FOUNTAINS are formed by the help of a condenser, which throws any quantity of air into a vessel in part filled with water; which by its elasticity forces the water up into pipes from which it is conveyed at pleasure.

The AIR-GUN is an instrument, in the form of a gun, by which a quantity of condensed air is suddenly set free, and drives a ball through the barrel with great force.

#### VI. The AIR-PUMP.

This instrument, the use of which is to exhaust the air from any vessel, has two strong barrels; within each of which, near the bottom, is fixed a valve opening upwards, and two pistons, one in each barrel, having a valve which likewise opens upwards. These pistons are moved by means of a cog-wheel, to the axis of which the handle is fixed, and whose teeth catch in the racks of the pistons, and move them upwards or downwards. When the handle is turned, one of the pistons is raised, and a vacuum produced in its barrel. By means of a pipe, which passes from an orifice in the plate upon which the receiver,



receiver, or vessel to be exhausted, stands, to the part of the barrel beneath the lower valve, the air contained in the receiver, communicating with the barrel, raises the lower valve by its elastic spring, and expands into the vacuum. Thus a part of the air in the receiver is extracted. By turning the handle the contrary way, the same effect is produced in the other barrel; whilst the first piston, being depressed, the air which had passed from the receiver is compressed, and escapes through the valve in the piston. This operation is continued till the air is nearly exhausted from the receiver: for it can never be perfectly exhausted, since at each stroke only such a part of the air which remained is taken away, as is to the quantity before the stroke, as the capacity of the barrel, to that of the receiver, pipe and barrel, taken together.

EXP. Exhibit the several instruments described in this proposition.



From the first settlement of the city in 1630 to the present time, the history of Boston is a record of growth and progress. The city has been the seat of government, commerce, and industry, and has played a prominent part in the history of the United States. The city has been the birthplace of many of the great men of the country, and has been the scene of many of the great events of our history.

The city of Boston is situated on a peninsula, and is bounded by the harbor on the south and east, and by the city of Cambridge on the north. The city is divided into several wards, and is governed by a mayor and a city council. The city is one of the most important cities in the United States, and is the center of commerce and industry in the New England region. The city has a long and rich history, and is a city of many firsts.

The city of Boston is a city of many firsts. It was the first city in the United States to have a public library, and it was the first city to have a public park. It was the first city to have a public hospital, and it was the first city to have a public school system. The city has a long and rich history, and is a city of many firsts.



B O O K IV.

O F O P T I C S ;

O R,

THE LAWS OF LIGHT AND VISION.

C H A P. I.

O F L I G H T.

DEF. I. **L**IGHT is that which, proceeding from any body to the eye, produces the perception of *seeing*.

DEF. II. A *Ray of Light*, is any exceedingly small portion of light, as it comes from a luminous body.

DEF. III. A body which is transparent, or affords a passage for the rays of light, is called a *Medium*.

DEF. IV. Rays of light which, coming from a point, continually separate as they proceed, are called *Diverging Rays*.

DEF. V. Rays which tend to a common point are called *Converging Rays*. The divergency, or convergency, of rays is measured by the angle contained between the lines which the rays describe.

DEF. VI. Rays of light are *parallel*, when the lines which they describe are parallel.

S

DEF.



DEF. VII. A *Beam* of light, is a body of parallel rays ; a *Pencil* of rays, is a body of diverging or converging rays.

DEF. VIII. The point from which diverging rays proceed, is called the *radiant point* ; that to which converging rays are directed, is called the *focus*.

Plate 6.  
Fig. 1.

If the rays proceed from B, BD, BA, BC, BE, are diverging rays, and B is the radiant : if the rays tend towards B, DB, AB, &c. are converging rays, and B is the focus.

Fig. 2.

If the rays AC, BC, converge to the focus C, passing on from thence in a right line, they become diverging, and C becomes a radiant.

DEF. IX. A ray of light bent from a straight course in the same medium, is said to be *inflected*.

### PROPOSITION I.

Rays of light consist of particles of matter.

For they are capable of being inflected out of their course by attraction.

EXP. I. If a beam of light be admitted into a dark room through a small hole, and the edge of a knife be brought near the beam, the rays, which would otherwise have been in a straight line, will be inflected towards the knife. The edge of any other thin plate of metal, &c. produces the same effect.

2. The shadow of a small body, as a hair, a thread, &c. placed in a beam of the sun's light, will be much broader than it ought to be if the rays of light passed by these bodies in right lines.

3. A beam of light passing through an exceedingly narrow slit, will be split into two, and leave a dark space in the middle.

### P R O P. II.

Every visible body emits particles of light from its surface in all directions, which, passing without obstruction, move in right lines.

Wherever a spectator is placed with respect to a luminous body, every point of that part of the surface which is turned towards him is visible to him : they are therefore emitted in all directions, and those rays only are intercepted in their passage by an interposed



terposed object, which would be intercepted upon the supposition that the rays move in right lines.

EXP. 1. Let a portion of a beam of light be intercepted by any body: the shadow of that body will be bounded by right lines passing from the luminous body, and meeting the lines which terminate the opaque body.

2. A ray of light, passing through a small orifice into a dark room, proceeds in a straight line.

3. Rays will not pass through a bended tube.

SCHOL. Rays of light are properly represented by right lines.

### P R O P. III.

The rays of light move with great velocity.

This is proved by observations made on the satellites of the planet Jupiter, and on the aberration of the rays of light from the fixed stars, as will be shewn in treating upon Astronomy.

### P R O P. IV.

The particles of light are exceedingly small.

Otherwise their velocity would render their momentum too great to be endured by the eye without pain.

EXP. 1. If a candle be lighted, and there be no obstacle to obstruct the progress of its rays, it will fill all the space within two miles every way before it has lost the least sensible part of its substance.

2. Rays of light will pass without confusion through a small puncture in a piece of paper, from several candles in a line parallel to the paper, and form distinct images on a sheet of pasteboard placed behind the paper.

### P R O P. V.

The quantities of light, received from a luminous body upon a given surface, are inversely as the squares of the distances of the surface from the luminous body.

Let ABD, EFG, be two concentric spherical surfaces; of which let ELFI, AHBK, be two similar portions. Let the rays CE and CF, with the rest proceeding from the center C, fall upon the portion ELFI, and cover it: it is evident from inspection, that the same rays at the distance CH will cover the portion AHBK only; now these rays being the same in number at each place, will be as much thinner in the former, than they

Plate 6.  
Fig. 3.



they are in the latter, as  $ELFI$  is larger than  $AHBK$ ; but these spaces being similar portions of the surfaces of spheres, have the same ratio to each other, that the surfaces themselves have, that is, they are to each other as the squares of their radii  $CL, CH$ : the density of the rays is therefore inversely as the squares of these radii, or of their distances from the luminous point  $C$ .

Exp. The light, passing from a candle through a square orifice, will diverge as it proceeds, and will illuminate surfaces which will be to each other as the squares of their distances from the candle.

## P R O P. VI.

If the distance between rays diverging from different radiant points be the same, the distances of the radiant points are inversely as the divergency of the rays.

Plate 6.  
Fig. 4.

Let  $D$  and  $E$  be two different radiants; and let the rays diverging from  $D$  describe the lines  $DA, DB$ , and the rays diverging from  $E$  describe the lines  $EA, EB$ ; so that, at the points  $A$  and  $B$ , the distance between the former rays shall be the same with the distance between the latter, and let  $EC, DC$ , be the perpendicular distances of the radiants  $E, D$ . At the point  $E$  make the angle  $ZEC$  equal to  $ADC$ , which is half  $ADB$ : whence  $ZEC$  and  $ADC$  (El. V. 7.) have the same ratio to  $AEC$ . But if these angles are small, they are very nearly in the proportion of their sines  $ZC, AC$ . And because the angle  $ADC$  is equal to the angle  $ZEC$  (El. I. 28.)  $AD$  is parallel to  $ZE$ : and because these lines are parallel (El. I. 29.) the angles  $CAD, CZE$ , are equal: whence the two triangles  $ZEC, ADC$ , are equiangular, and (El. VI. 4.)  $EC$  is to  $DC$ , as  $ZC$  to  $AC$ , or (from what was shewn above) as  $ADC$  to  $AEC$ : that is, the distance of the radiant  $E$  is to the distance of the radiant  $D$ , as half the angle of divergency of the rays which proceed from  $D$  is to half the divergency of the rays which proceed from  $E$ , or as the whole angle of divergency  $ADB$  to the whole angle of divergency  $AEB$ ; that is, the distances of the radiants are inversely as the divergency of the rays.

## P R O P. VII.

If the distance between converging rays tending to different foci be the same, the distances of the foci are inversely as the convergency of the rays.

Plate 6.  
Fig. 4.

Let  $AD, BD$ , be lines described by rays converging to the focus  $D$ , and  $AE, BE$ , lines described by other rays converging to  $E$ , and let the distance  $AB$ , at the points  $A$  and



and B, be the same between the former and the latter rays. The angles ADB, AEB, are in this case the angles of convergency; and EC, DC, are distances of the foci to which they respectively tend. Now, it was proved in the last Prop. that EC is to DC as ADB to AEB. Therefore the distances of the foci are inversely as the convergency of the rays.

## P R O P. VIII.

If rays proceed from a radiant at an infinite distance, their divergency is nothing, and the rays are considered as parallel.

Since (by Prop. VI.) the divergency of rays is inversely as the distance of the radiant, when the distance of the radiant is infinitely great, the angle of divergency is infinitely small, and the rays may be considered as parallel.

COR. Hence all the rays which come from the center, or any other given point, of the sun's surface, are considered as parallel.

## P R O P. IX.

If rays tend to a focus at an infinite distance, their convergency is nothing, and the rays are considered as parallel.

Since (by Prop. VII.) the convergency is inversely as the distance of the focus, when that distance is infinitely great, the angle of convergency is infinitely small.



## C H A P. II.

## O F R E F R A C T I O N.

## S E C T. I.

*Of the LAWS of REFRACTION.*

DEF. X. A ray of light bent from a straight course by passing out of one medium into another, is said to be *refracted*.

DEF. XI. The *Angle of Incidence*, is that which is contained between the line described by the incident ray, and a line perpendicular to the surface on which the ray strikes, raised from the point of incidence.

DEF. XII. The *Angle of Refraction*, is that which is contained between the line described by the refracted ray, and a line perpendicular to the refracting surface at the point in which the ray passes through that surface.

DEF. XIII. The *Angle of Deviation*, is that which is contained between the line of direction of an incident ray, and the direction of the same ray after it is refracted.

Plate 6.  
Fig. 6.

AC is a ray of light; HK the surface of the refracting medium; CF the refracted ray; OP the perpendicular; ACO the angle of incidence; PCF the angle of refraction, and FCL the angle of deviation.

Plate 6.  
Fig. 5.

SCHOL. The radiant point and focus may be either real or imaginary. If the rays  $rn$ ,  $ro$ , diverging from the radiant  $r$ , pass into a refracting medium, and move on in the directions of the lines  $nA$ ,  $oB$ , which produced in the contrary direction would meet in  $R$ , this radiant point is imaginary.

If the rays  $Ip$ ,  $Lq$ , tending towards the point  $F$ , be refracted at  $p$  and  $q$ , and acquire a direction towards  $f$ , the focus  $f$  is imaginary.

PROP.



## P R O P. X.

The attracting force of any medium, acting upon a ray of light, is every where perpendicular to the refracting surface.

If the medium be uniform in all its parts, its immediate power upon the ray of light will be equally strong in every point of a plane drawn parallel to the refracting surface; though its strength may be different in the next parallel plane, and so onwards as far as that power is extended on each side of the surface of the medium. The extent of this power will therefore be terminated by two planes, parallel to one another and to the refracting surface. Let  $R$  be a particle of light, acted upon by the refractive power of the medium whose refracting surface is  $DC$ . It is evident that the refractive power at  $O$  will move the particle  $R$  in the direction  $RO$ : and taking any two points  $D, C$ , at equal distances on each side of  $O$ , the powers at  $D$  and  $C$  being equal, and acting at equal distances,  $RD, RC$ , equally inclined to  $RO$ , cannot move  $R$  in any direction but that of  $RO$ . The same may be shewn of the powers at every point of the line  $DC$ , and in every line parallel to  $DC$ , that is of the whole power of the medium.

Plate 6.  
Fig. 8.

## P R O P. XI.

A ray of light, in passing out of a rarer into a denser medium, is refracted towards a perpendicular to the surface of the denser, raised from the point in which the ray meets the medium: in passing out of a denser into a rarer medium, it is refracted from the same perpendicular.

Let a ray of light,  $AC$ , pass obliquely out of a rarer medium  $X$ , into a denser medium  $Z$ ; let  $HK$  be the plane surface of the denser medium; from the point  $C$ , in which the ray  $AC$  passes into the denser medium, raise the perpendicular  $OCP$ : the ray will be refracted out of the direction  $ACL$ , towards the perpendicular  $OCP$ .

Plate 6.  
Fig. 6.

Because the ray is more attracted by the denser medium than by the rarer, it will be accelerated on entering the medium  $Z$ : for whilst the ray is so near the surface of the medium  $Z$  as to be within its attraction, and more attracted downwards than upwards, this attraction conspires with the motion of the ray, and consequently increases its velocity. And, since the action of the attracting force of the medium  $Z$ , must (by Prop. X.) be in the direction of a line  $OCP$  perpendicular to its surface, if the oblique motion of the ray in the direction  $AC$  be resolved into two others,  $AD$  parallel to the surface  $HK$ , and  $AB$ , or  $DC$ , perpendicular to it, the parallel motion  $AD$  cannot  
be



be accelerated or retarded by the attraction which acts in the direction  $OC$ : the change of velocity therefore which the ray receives from the attracting force, must be made in the perpendicular part of its motion  $DC$ . Take  $CG$  greater than  $DC$  representing the perpendicular motion of the ray after passing into the denser medium; and take  $CE$  equal to  $AD$  representing the parallel part of the motion of the ray, which, because it is parallel to  $AB$ , remains the same when the ray enters the denser medium. The ray therefore, at its entering the medium  $Z$ , may be considered as acted upon by two forces  $CE$ ,  $CG$ , and consequently (Book II. Prop. XIV.) will describe the diagonal  $CF$  of a parallelogram, the sides of which are  $CE$ ,  $CG$ . Now, of these sides  $CE$  remaining the same, whilst  $CG$  becomes greater than  $CD$ , the angle  $GCF$  (from the nature of the parallelogram) will be less than the angle  $NCL$ , equal (El. I. 15.) to  $ACD$ . Therefore the ray after it has passed into the denser medium, makes a less angle with the perpendicular  $OCP$  than  $AC$ , the ray before it passes into the denser medium; that is, the ray in passing out of the rarer into the denser medium, is refracted towards the perpendicular. On the contrary, whilst the ray of light  $FC$  is passing out of the denser medium  $Z$  into the rarer medium  $X$ , it is more attracted by the denser than by the rarer medium, and is therefore more drawn downwards than upwards: whence the attraction opposes the motion of the ray, and will retard it as much as in passing out of the rarer into the denser medium it was accelerated: and consequently, the effect will be the reverse of that which was shewn in the former case.

Plate 6.  
Fig. 7.

EXP. I. Let a perpendicular cylindrical vessel be so placed that the sun, shining upon its side  $NA$ , may cast the shadow of the side to a point  $L$  in the bottom of the vessel. This shadow is terminated by  $SNL$ , a ray which passes, in a right line, by the edge of the vessel. If the vessel be filled with water, the shadow will recede, as the water is poured into the vessel, from the point  $L$  which terminated it when the vessel was empty, towards the side  $NA$ , on which the sun shines, and will be terminated by the ray  $ONC$ : that is, the ray  $SNL$ , which first terminated the shadow, by passing out of the air into the water, is refracted towards  $AN$ , a line drawn perpendicular to the surface of the water, at the point in which the ray enters the water: or the angle of refraction is less than the angle of incidence.

2. Let a small bright object be laid upon the bottom of a cylindrical vessel  $NBAL$  at  $C$ . Let the spectator's eye be so placed at  $S$ , as just to lose sight of the object at  $C$ , that is, so that a ray passing in a right line from the remote edge of the object towards the eye at  $S$  will be intercepted by the edge of the vessel, or that the first ray which is not intercepted passes in the direction  $ONC$  above the eye. Whilst the eye continues in the same situation, if the vessel be filled with water, the object will become visible; that is, the ray which passed from the remote edge of the object, in a right line  $CNO$ , by the vessel, in entering the air is refracted into the direction  $NS$ , towards the eye, or from the perpendicular  $PNA$ .

P R O P.



## P R O P. XII.

All refraction is reciprocal.

The ray AC, in passing out of the medium X into Z, is refracted into CF, because it is accelerated at its entrance into Z by the greater attraction of the denser medium: and the ray FC in passing out of Z into X is refracted into CA, because it is retarded by the same attraction. Since then the acceleration and retardation are produced by the same degree of attraction in opposite directions, they will be equal to one another, and the refractions produced by them will be equal, but in opposite directions: that is, if the refracted ray becomes the incident ray, the incident ray will become the refracted one; or, the refractions are reciprocal.

Plate 6.  
Fig. 6.

## P R O P. XIII.

In any two given mediums, the sine of any one angle of incidence has the same ratio to the sine of the corresponding angle of refraction, as the sine of any other angle of incidence has to the sine of its corresponding angle of refraction.

The oblique motion of the ray AC, in passing out of the rarer medium X into the denser medium Z, may be resolved into AD parallel to the surface HK, and AB or DC perpendicular to it. Of these (as was shewn in Prop. XI.) only the perpendicular motion DC is accelerated. Take CN equal to CD, and continue the refracted ray CS to F, so that if FG is drawn perpendicular to OP, FG may be equal to AD; and draw NS the sine of the angle of refraction NCS. The perpendicular velocity of the ray in the medium Z (by Prop. XI.) is CG, that is, the ray is accelerated in the ratio of CN to CG. But, because CGF, CNS, are equiangular triangles, (El. VI. 4.) GF the sine of the angle of incidence is to NS the sine of the angle of refraction, as CG the velocity of the ray in the denser medium Z, is to CN the velocity of the ray in the rarer medium X; that is, the sines of the angles of incidence are inversely as the velocities of the ray. But where the two mediums are given, as X and Z, their densities, and consequently the differences of their attractions are given: and, since the ratio of the velocities in different mediums depends upon the difference of their attracting force, where this difference is given, the ratio of the velocities cannot be altered by changing the obliquity of the ray, so that, if the velocity in one medium is to that in another, as CG to CN, and the sine of the angle of incidence be to that of refraction in any one degree of obliquity, as GF to NS, this ratio will continue the same in every degree of obliquity; for GF will always be to NS in the ratio of CG to CN.

Plate 6.  
Fig. 6.

T

COR.



COR. 1. Hence, when the angle of incidence is increased, the corresponding angle of refraction will also be increased: because the ratio of their sines cannot continue the same, unless they be both increased: and, if two angles of incidence be equal, the angles of refraction will also be equal.

COR. 2. Hence the angle of deviation varies with the angle of incidence.

Plate 6.  
Fig. 6.

SCHOL. If a ray of light, AC, pass obliquely out of air into glass, AD the sine of the angle of incidence ACD, is to NS the sine of the angle of refraction NCD, nearly as 3 to 2: therefore, supposing the sines proportional to the angles, the sine of NCL the angle of deviation, is as the difference between AD and NS, that is, as  $3-2$ , or 1, whence the sine of incidence is to the sine of the angle of deviation as 3 to 1. In like manner it may be shewn, that, when the ray passes obliquely out of glass into air, the sine of the angle of incidence will be to that of deviation, as NS to  $AD-NS$ , that is, as 2 to 1. In passing out of air into water, the sine of the angle of incidence is to that of refraction, as 4 to 3, and to that of deviation, as 4 to  $4-3$ , or 1: and in passing out of water into air, the sine of the angle of incidence is to that of refraction, as 3 to 4, and to that of deviation, as 3 to 1.

Plate 6.  
Fig. 6.

EXP. Upon a smooth board, about the center C, describe a circle HOKP; draw two diameters of the circle OP, HK, perpendicular to each other; draw ADM perpendicular to OP; cut off DT and CI equal to  $\frac{3}{4}$  DA; through TI, draw TIS, cutting the circumference in S; NS drawn from S perpendicularly upon OP, will be equal to DT, or  $\frac{3}{4}$  of DA. Then if pins be stuck perpendicularly at A, C and S, and the board be dipped in the water as far as the line HK, the pin at S will appear in the same line with the pins at A and C. This shews, that the ray which comes from the pin S is so refracted at C, as to come to the eye along the line CA: whence the sine of incidence AD is to the sine of refraction FE, as 4 to 3. If other pins were fixed along CS, they would all appear in AC produced; which shews that the ray is bent at the surface only. The same may be shewn, at different inclinations of the incident ray, by means of a moveable rod turning upon the center C, which always keep the ratio of the sines AD, NS, as 4 to 3. Also the sun's shadow, coinciding with AC, may be shewn to be refracted in the same manner.

#### P R O P. XIV.

Rays of light which pass perpendicularly out of one medium into another, suffer no refraction.

Plate 6.  
Fig. 6.

When AC, the incident ray, coincides with OC, the perpendicular, the action of the medium Z or X to accelerate or retard the motion of the ray, being perpendicular to its surface, cannot turn the ray out of its perpendicular path.

P R O P.



## P R O P. XV.

When parallel rays pass obliquely out of one medium into another through a plane surface, they will continue parallel after refraction.

Let AB, CD, be parallel rays, falling on the plane surface RBD of a medium of different density : because they make equal angles of incidence with their respective perpendiculars OP, ST, they will suffer an equal degree of refraction ; that is, the angles of refraction, EBP, FDT, will be equal ; whence the refracted rays BE, DF will be parallel.

Plate 6.  
Fig. 9.

## P R O P. XVI.

Through a plane surface, if diverging rays pass out of a rarer into a denser medium, they are made to diverge less ; and if they pass out of a denser into a rarer medium, they are made to diverge more : if converging rays pass out of a rarer into a denser medium, they will be made to converge less ; if out of a denser into a rarer, to converge more.

Let the diverging rays AB, AE, AF, pass out of a rarer into a denser medium, through the plane surface GH, and let the ray AB be perpendicular to that surface ; the rest being refracted towards their respective perpendiculars IK, LM, and that the most which falls the farthest from B, they will proceed in the directions EN and FO, diverging in a less degree from the ray AP, than they did before refraction : whereas had they proceeded out of a denser into a rarer medium, they would have been refracted from their perpendiculars EK, FM, and those the most which were the most oblique, and therefore would have diverged more than before. Again, let the converging rays AB, CD, EF, pass out of a rarer into a denser medium, through the plane surface GH, and let the ray AB be perpendicular to that surface ; the other rays being refracted towards their respective perpendiculars IK, LM, and EF being refracted more than CD, they will proceed in the directions DN, FN, converging in a less degree towards the ray AN, than they did before : whereas, had the first medium been the denser, they would have been refracted the other way, and therefore have converged more.

Plate 6.  
Fig. 10.

Fig. 11.

DEF. XIV. A *Lens* is a round piece of polished glass, which has both its sides spherical, or one spherical, and the other plane.

A lens may either be convex on both sides, plano-convex, concave, plano-concave, or convex on one side, and concave on the other ; which last is called a *meniscus*.



In plate 6. fig. 12. Sections of these, formed by a plane passing perpendicularly through their centers, are represented.

DEF. XV. The *Axis of a Lens*, is a right line, passing through its center, perpendicular to both its surfaces, and the extremities of the axes are its poles.

Each kind of lens is generated by the revolution of a section of the lens about this line. Thus, in the first lens, if *acb*, *adb*, revolve about *cd*, the convex lens will be formed.

DEF. XVI. In every beam of light, the middle ray is called the *Axis*.

DEF. XVII. Rays are said to fall *directly* upon a lens, if their axis coincides with the axis of the lens; otherwise, they are said to fall *obliquely*.

DEF. XVIII. The point in which parallel rays are collected by passing through a lens, is called the *Focus of parallel rays* of that lens.

#### P R O P. XVII.

Through a convex surface of the denser medium, parallel rays, passing out of a rarer into a denser medium, will become converging; diverging rays will be made to diverge less, to become parallel, or to converge, according to the degree of divergency before refraction, or of the convexity of the surface; rays converging towards the center of convexity, will suffer no refraction; rays converging to a point beyond the center of convexity, will be made more converging; and rays converging towards a point nearer the surface than the center of convexity, will be made less converging by refraction: and when the rays proceed out of a denser into a rarer medium, the contrary occurs in each case.

Plate 6.  
Fig. 13.

Let *AB*, *ID*, be parallel rays entering a denser medium through the convex surface *CDE*, whose center of convexity is *L*; and let one of these, *ID*, be perpendicular to the surface. This will pass on through the center without suffering any refraction, but the other being oblique to the surface, will be refracted towards the perpendicular *LB*, and will



will therefore be made to proceed in some line, as PK, converging towards the other ray, and meeting it in K, the focus. Had one ray diverged from the other, suppose in the line MP, it would, by being refracted towards its perpendicular LB, have been made either to diverge less, be parallel, or to converge. Let the line ID be produced to K; and if the ray had converged, so as to have described the line NBL, it would then have been coincident with its perpendicular, and have suffered no refraction. If it had proceeded with less convergency towards any point beyond L in the line IK, it would have been made to converge more by being refracted towards the perpendicular LB, which converges more than it; and had it proceeded with more convergency than BL, that is, towards any point between D and L, being refracted towards the perpendicular, it would have been made to converge less.

And the contrary happens, when rays proceed out of a denser into a rarer medium, through a concave surface of the denser. For being now refracted from their respective perpendiculars, as they were before towards them, if they are parallel before refraction, they diverge afterwards; if they diverge, their divergency is increased; if they converge in the direction of their perpendiculars, they suffer no refraction; if they converge less than their respective perpendiculars, they are made to converge still less, to be parallel, or to diverge; if they converge more, their convergency is increased. All which may clearly be seen by the figure, imagining the rays AB, ID, &c. bent the contrary way in their refractions to what they were in the former cases.

EXP. Let parallel, diverging, and converging rays, pass through a convex lens; the several cases of this proposition will be confirmed.

If CDEH be a convex lens, whose axis is IK, let L be the center of the first convexity CDE, and M that of the other CHE; and let the ray AB be parallel to the axis; through B draw the line LN, which will be perpendicular to the surface CDE at that point. The ray AB in entering the denser substance of the lens will be refracted towards the perpendicular, and therefore proceed after it has entered the surface at B in some direction inclined towards the axis, as BP. Through M the center of convexity of this surface and the point P draw the line MR, which passing through the center will be perpendicular to the surface at P, and the ray now entering a rarer medium will be refracted from the perpendicular into some direction as PK. In like manner, and for the same reasons, the parallel ray ST on the other side the axis, and also all the intermediate ones, as XZ, &c. will meet it in the same point, unless the rays AB and ST enter the surface of the lens at too great a distance from the axis IK, the reason of which will be afterwards explained.

The point K where the parallel rays AB, ST, &c. are supposed to be collected by passing through the lens CE, is called the focus of parallel rays of that lens.

If the rays come diverging from a point equally distant from the surface as the focus of parallel rays, they will be rendered parallel; if from a point farther from the surface than L, they will be brought to a point beyond L; if from a point nearer than L, they will diverge less; as may be inferred from Prop. XII.



If the rays come converging towards  $L$ , they will suffer no refraction; if towards a point beyond  $L$ , they will become more converging; if towards a point nearer the surface than  $L$ , they will become less converging: as is sufficiently explained in the proof of this proposition.

Plate 6.  
Fig. 14.

SCHOL. If the rays  $AB$ ,  $CD$ ,  $EF$ , be parallel to each other, but oblique to  $GH$  the axis of the lens  $IK$ , or if the diverging rays  $CB$ ,  $CF$ , proceed as from some point  $C$  which is not situated in the axis of the lens, they will be collected into some point as  $L$ , not directly opposite to the radiant  $C$ , but nearly so: for the ray  $CD$ , which passes through the middle of the lens, and falls upon the surface of it with some obliquity, will itself suffer a refraction at  $D$  and  $N$ ; but it will be refracted the contrary way in one place from that in the other, and these refractions will be equal in degree if the lens has an equal convexity on each side, as we may easily perceive if we imagine  $ND$  to be a ray passing out of the lens both at  $N$  and  $D$ , for it is evident the line  $ND$  has an equal inclination to each surface at both its extremities. Upon which account the difference between the situation of the point  $L$ , and one directly opposite to  $C$ , is so small, that it is generally neglected; and the focus is supposed to be in that line, in which a ray, that would pass through the middle point of the lens, were it to suffer no refraction, would proceed.

#### P R O P. XVIII.

When rays pass out of a rarer into a denser medium, through a concave surface of the denser, if the rays are parallel before refraction, they are made to diverge: if they are divergent, they are made to diverge more, to suffer no refraction, or to diverge less, according as they proceed from some point beyond the center, from the center, or from some point between the center and the surface: if they are convergent, they are either made less converging, parallel, or diverging, according to their degree of convergency before refraction: and the reverse, in passing out of a denser into a rarer medium.

Plate 6.  
Fig. 15.

Let  $MF$ ,  $OI$ , be two parallel rays entering a concave and denser medium, the center of whose convexity is  $H$ , and the perpendicular to the refracting surface at the point  $F$  is  $LH$ ; the ray  $OI$ , if we suppose it perpendicular to the surface, will proceed on directly without refraction, but the oblique ray  $MF$ , being refracted towards the perpendicular  $HL$ , will recede from the other ray  $OI$ . If the ray  $MF$  had proceeded from a point in  $IO$  farther from the surface than  $H$ , it would have been bent nearer to the perpendicular, and therefore



therefore have diverged more: if it had diverged from the center  $H$ , it would have fallen in with the perpendicular  $HL$ , and not have been refracted at all: and had it proceeded from a point nearer the surface than the center  $H$ , it would by being refracted towards the perpendicular  $HL$ , have proceeded in some line nearer it than it otherwise would have done, and so would diverge less than before refraction. Lastly, if it had converged, it would have been rendered less converging, parallel, or diverging, according to the degree of convergency, which it had before it entered into the refracting surface.

If the same rays proceed out of a denser into a rarer medium through a convex surface of the denser, the contrary happens in each supposition: the parallel rays are made to converge; those which diverge less than their respective perpendiculars, that is, those which proceed from a point beyond the center, are made less diverging, parallel, or converging, according to the degree in which they diverge before refraction; those which diverge more than their respective perpendiculars, that is, those which proceed from a point between the center and the refracting surface, are made to diverge still more. And those which converge, are made to converge more. All which may easily be seen by considering the situation of the rays with respect to the perpendicular  $HL$ .

EXP. Let parallel, diverging, and converging rays pass through a concave lens: the several cases of this proposition will be confirmed: thus, let  $ABCD$  represent a concave lens,  $EO$  its axis,  $FH$  the radius of the first concavity,  $IK$  that of the second; produce  $HF$  to  $L$ , and let  $MF$  be a ray of light entering the lens at the point  $F$ . This ray being refracted towards the perpendicular  $FL$ , will pass on to some point, as  $K$  in the other surface, more distant from the axis than  $F$ , and being there refracted from the perpendicular  $IK$ , will be diverted farther still from the axis, and proceed on the direction  $KN$ , as from some point  $O$ , on the first side of the lens. In like manner other rays, as  $PQ$  parallel to the former, will proceed after refraction at both surfaces as from the same point  $O$ ; which upon that account will be the imaginary focus of parallel rays of this lens.

Plate 6.  
Fig. 15.

If the rays diverge before they enter the lens, their imaginary focus is then nearer the lens than that of the parallel rays. If they converge before they enter the lens proceeding towards some distant point in the axis, as  $E$ , they are then rendered less converging: if they converge to a point at the same distance from the lens with the focus of parallel rays, they then go out parallel; if to a point at a less distance, they remain converging, but in a less degree than before they entered the lens.

SCHOL. If the lens is plane on one side, and convex or concave on the other, the refraction is similar, but in a less degree. In a meniscus, if the convexity on one side is equal to the concavity on the other, the two sides will produce equal and contrary effects, and the inclination of the rays to each other will be the same after refraction as before. If the convexity is greater than the concavity, the meniscus will have the effect of a lens which has its convexity equal to the excess of the convexity of the meniscus above its concavity; and the reverse, if its concavity exceeds its convexity.

PROP.



## P R O P. XIX.

When diverging rays are made to converge by passing through a convex lens, as the radiant approaches towards the lens on one side, the focus departs from it on the other; and the reverse.

For, the nearer the radiant point is to the lens, the more the rays which fall upon the lens diverge before refraction; whence (the power of the refracting medium being given) they will converge the less after refraction, and have their focal point at the greater distance from the surface: on the contrary, the more remote the radiant point is from the lens, the less the incident rays will diverge, and consequently the more will the refracted rays converge, and the nearer will the focus be to the surface, till, at an infinite distance of the radiant, the rays are collected in the focus of parallel rays.

## P R O P. XX.

When the radiant point is at that distance from the surface, at which parallel rays coming through it from the other side would be collected, rays flowing from that point become parallel on the other side.

Plate 6.  
Fig. 13.

It manifestly follows from Prop. XII. that if the parallel rays AB, ID, ST, in passing through CDE, are brought to a focus in K, rays from K as a radiant point will, after refraction, proceed in the parallel lines BA, DI, TS.

## P R O P. XXI.

When rays pass out of one medium into another of different density through a plane surface, if they diverge, the focal distance will be to that of the radiant point; if they converge, it will be to that of the imaginary focus of the incident rays, as the sine of the angle of incidence is to that of the angle of refraction.

This proposition admits of four cases.

Case 1. Of diverging rays passing out of a rarer into a denser medium.

Plate 6.  
Fig. 16.

Let X represent a rarer, and Z a denser medium, separated from each other by the plane surface AB; suppose CD and CE to be two diverging rays proceeding from the point C, the one perpendicular to the surface, the other oblique; through E draw the perpendicular



perpendicular PK. The ray CD being perpendicular to the surface, will proceed on in the right line CQ, but the other falling upon it obliquely at E, and there entering a denser medium, will suffer a refraction towards the perpendicular EK. Let then EG be the refracted ray, and produce it back till it intersects DC produced also in F; this will be the focal point. On the center E with the radius EF, describe the circle AFBQ, and produce EC to H; draw HI the sine of the angle of incidence, and GK that of refraction; equal to this is FP or CM, which let be drawn. Now if we suppose the points D and E contiguous, or nearly so, then will the line HE be almost coincident with FD, and therefore FD will be to CD, as HE to CE, or (El. VI. 4.) as HI to CM; that is, the focal distance of the ray CE is to the distance of the radiant point, as the sine of the angle of incidence is to that of the angle of refraction.\*

Plate 6.  
Fig. 16.

Case 2. Of diverging rays proceeding out of a denser into a rarer medium.

Let X be the denser, Z the rarer medium, FD and FE two diverging rays proceeding from the point F; and supposing the perpendicular PK drawn as before, FP will be the sine of the angle of incidence of the oblique ray FE, which in this case being refracted from the perpendicular, will pass on in some line as ER, which being produced back to the circumference of the circle will cut the ray FD somewhere, suppose in C, this therefore will be the imaginary focus of the refracted ray ER; draw RO the sine of the angle of refraction, to which HI will be equal: but here also FP or its equal CM, is to HI, as EC to EH, or (if the point D and E be considered as contiguous) as DC to DF; that is, the sine of the angle of incidence is to the sine of the angle of refraction, as the focal distance to that of the real radiant point.

\* Whereas IE is to ME, or ND to CD, as HI to CM, that is, as the ratio of the sine of the angle of incidence to that of the angle of refraction, which (Prop. XIII.) is always the same, the line IN is in all inclinations of the ray CE, at the same distance from CM. Consequently, had CE been coincident with CD, the point H had fallen upon N; and because the circle passes through both H and F, F would also have fallen upon N; upon which account the focus of the ray CE would have been there. But the ray CE being oblique to the surface DB, the point H is at some distance from N; and therefore the point F is necessarily so too, and the more so, the greater that distance is. Hence it is manifest, that no two rays flowing from the radiant point C, and falling with different obliquities on the surface BD, will after refraction there, proceed as from the same point; therefore strictly speaking, there is no one point in the line D produced, that can more properly be called the focus of rays flowing from C, than another: for those which enter the refracting surface near D, will, after refraction, proceed, as has been observed, from the parts about N; those which enter near E, will flow as from the parts about F; those which enter about T, as from some points in the line DF produced, &c. And it is farther to be observed, that when the angle DCE becomes large, the line NF increases apace; whence those rays which fall near T, proceed, after refraction, as from a more diffused space, than those which fall at the same distance from each other near the point D. Upon which account it is usual with optical writers to suppose the distance between the points where the rays enter the plane surface of a refracting medium, to be inconsiderable with regard to the distance of the radiant point, if they diverge; or to that of their imaginary focus, if they converge: and unless there be some particular reason to the contrary, they consider them, as entering the refracting medium in a direction as nearly perpendicular to its surfaces as may be.



Plate 6.  
Fig. 16.

Case 3. Of converging rays passing out of a denser medium into a rarer.

Next; Let  $Z$  be the denser,  $X$  the rarer medium, and  $GE$  the incident ray; this will be refracted from the perpendicular into a line as  $EH$ ; then all things remaining as before,  $GK$ , or its equal  $FP$ , or  $CM$  will be the sine of the angle of incidence, and  $HI$  that of refraction: but these lines, as before, are to each other, as  $DC$  to  $DF$ ; that is, the focal distance is to the distance of the imaginary focus, as the sine of the angle of incidence to that of the angle of refraction.

Case 4. Of converging rays passing out of a rarer into a denser medium.

Let  $Z$  be the rarer,  $X$  the denser medium, and  $RE$  the incident ray; this will be refracted towards the perpendicular into a line, as  $EF$ ;  $C$  will be the imaginary focus, and  $F$  the real one,  $HI$  which (El. I. 26.) is equal to  $RO$ , will be the sine of the angle of incidence, and  $FP$  that of the angle of refraction: but these are to each other (El. VI. 2.) as  $DF$  to  $DC$ ; and therefore the focal distance is to that of the imaginary focus, as the sine of the angle of incidence is to that of the angle of refraction.

## P R O P. XXII.

When parallel rays fall upon a spherical surface of different density, the focal distance will be to the distance of the center of convexity, as the sine of the angle of incidence is to the difference between that sine and the sine of the angle of refraction.

Case 1. Of parallel rays passing out of a rarer into a denser medium through a convex surface of the denser.

Plate 6.  
Fig. 17.

Let  $AB$  represent a convex surface;  $C$  its center of convexity;  $HA$  and  $DB$  two parallel rays, passing out of a rarer medium into a denser, the one perpendicular to the refracting surface, the other oblique: draw  $CB$ ; this being a radius, will be perpendicular to the surface at the point  $B$ ; and the oblique ray  $DB$  being in this case refracted towards the perpendicular, will proceed in some line, as  $BF$ , meeting the other ray in  $F$ , which will therefore be the focal point; produce  $CB$  to  $N$ , then will  $DBN$ , or (El. I. 29.) its equal  $BCA$  be the angle of incidence, and  $FBC$  that of refraction. Now, since any angle has the same sine with its complement to two right ones, the angle  $FCB$  being the complement of  $ACB$ , which is equal to the angle of incidence, may here be taken for that angle; and therefore, as the sides of a triangle have the same relation to each other, as the sines of their opposite angles have,  $FB$  being opposite to this angle, and  $FC$  being opposite to the angle of refraction, they may here be considered as the sines of the angles of incidence and of refraction; and for the same reason  $CB$  may be considered as the sine of the angle  $CFB$ , which angle, being, together with the angle  $FBC$ , equal to the external one  $ACB$  (El. I. 32.) it is itself equal to the difference between those two last angles;



angles; and therefore the line FB is to CB, as the sine of the angle of incidence is to the sine of an angle which is equal to the difference between the angle of incidence and of refraction. Now because in very small angles as these are, for we suppose in this case also the distance AB to vanish, (the reason of which will be shewn in the note,) their sines have nearly the same ratio to each other that they themselves <sup>have</sup>, the distance FB will be to CB as the sine of the angle of incidence is to the difference between that sine and the sine of the angle of refraction: but because BA vanishes, FB and FA are equal, and therefore FA is to CA in that ratio.\*

Case 2. Of parallel rays passing out of a denser into a rarer medium through a concave surface of the denser.

Let AB be the concave surface of the denser medium, C the center of convexity, and HA and DB two parallel rays. Through B the point where the oblique ray DB enters the rarer medium, draw the perpendicular CN; and let the ray DB, being in this case refracted from the perpendicular, proceed in the direction BM; produce BM back to H; this will be the imaginary focus, and DBN, or its equal ACB, will be the angle of incidence, and CBM, or its equal (El. I. 15.) HBN that of refraction; produce DB to L, and draw BF such, that the angle LBF may be equal to DBH: then because NBD and DBH together are equal to NBH the angle of refraction, therefore BCA which is equal to the first, and LBF which is equal to the second, are together equal to the angle of refraction; but LBF is equal to BFA (El. I. 29.) consequently, BFA and BCA together are equal to the angle of refraction; and therefore since one of them, BCA, is equal to the angle of incidence, the other, BFA, is the difference between that angle, and the angle of refraction. Now FB the sine of the angle FCB, or which is the same thing, of its complement to two right ones, BCA, the angle of incidence, is to CB the sine of the angle BFC, as FB to CB, that is, as HB to CB; for the angles DBH and LBF being equal, the lines BF and BH are so too; but the distance BA vanishing, HB is to CB, as HA to CA: that is, the sine of the angle of incidence is to the sine of an angle which is the difference between the angle of incidence and refraction,

Plate 6.  
Fig. 17.

\* It appears from the above proposition, that the focal distance of the oblique ray DB, is such, that the line BF shall be to the line CB or CA as the sine of the angle of incidence to the sine of an angle equal to the difference between the angle of incidence and refraction; therefore so long as the angles BCA, &c. are small, so long the line FB is nearly of the same length, because small angles have nearly the same ratio to each other that their sines have. But when the point B is removed far from A, so that the ray DB enters the surface, suppose about O, the angles BCA, &c. becoming large, the sine of the angle of incidence begins to bear a considerably less ratio to the sine of an angle which is equal to the difference between the angle of incidence and refraction than before, and therefore the line BF begins to bear a much less ratio to BC; wherefore its length decreases apace: upon which account those rays which enter the surface about O, not only meet nearer the center of convexity than those which enter at A, but are collected into a more diffused space. From hence it is, that the point where those only which enter near A, are collected, is reckoned the true focus; and the distance AB in all demonstrations relating to the foci of parallel rays entering a spherical surface, whether convex or concave, is supposed to vanish.



Plate 6.  
Fig. 17.

or because the angles are small, to the difference between the sine of the angle of incidence and that of refraction, as the distance of the focus from the surface is to that of the center from the same.

Case 3. Of parallel rays passing out of a rarer into a denser medium through a concave surface of the denser.

Let AB be the concave surface of the denser medium, and let LB and FA be the incident rays. Now whereas, when DB was the incident ray, and passed out of a rarer into a denser medium, as in Case the first, it was refracted into a line BF, this ray LB having the same inclination to the perpendicular, will also suffer the same degree of refraction, and will therefore pass on afterwards in the line FB produced, towards P. So that, whereas in that case the point F was the real focus of the incident ray DB, the same point will, in this case, be the imaginary focus of the incident ray LB: but it was there demonstrated, that the distance FA is to CA, as the sine of the angle of incidence is to the difference between that and the sine of the angle of refraction, therefore the focal distance of the refracted ray BP is to the distance of the center of convexity in that ratio.

Case 4. Of parallel rays passing out of a denser into a rarer medium through a convex surface of the denser.

Let AB be the convex surface of the denser medium, and let LB and FA be the incident rays, as before. Now whereas, when DB was the incident ray passing out of a denser into a rarer medium, it was refracted into BM, as in Case the second, having a point as H in the line MB produced for its imaginary focus; therefore LB, for the like reason as was given in the last case, will in this be refracted into BH, having the same point H for its real focus. So that here also the focal distance will be to that of the center of convexity, as the sine of the angle of incidence is to the difference between that and the sine of the angle of refraction.

COR. Hence, the sines of the angles of incidence and of refraction of parallel rays being given, and also the distance of the center of convexity from the surface, the focus of any lens may be easily found.

### P R O P. XXIII.

When diverging or converging rays enter into a medium of different density through a spherical surface, the ratio compounded of that which the focal distance bears to the distance of the radiant point (or of the imaginary focus of the incident rays, if they converge) and of that, which the distance between the same radiant point (or imaginary focus) and the center, bears to the distance between the  
center



center and the focus, is equal to the ratio, which the sine of the angle of incidence bears to the sine of the angle of refraction.

Case I. Of diverging rays passing out of a rarer into a denser medium through a convex surface of the denser, with such a degree of divergency, that they shall converge after refraction.

Let BD represent a spherical surface, and <sup>C</sup> its center of convexity; and let there be two diverging rays AB and AD, proceeding from the radiant point A, the one perpendicular to the surface, the other oblique. Through the center C produce the perpendicular ray AD to F, and draw the radius CB and produce it to K, and let BF be the refracted ray; then will F be the focal point: produce AB to H, and through the point F draw the line FG parallel to CB. AB being the incident ray, and CK perpendicular to the surface at the point B, the angle ABK, or which is equal to it, because of the parallel lines CB and FG, FGH is the angle of incidence. Now, since the complement of any angle to two right ones has the same sine with the angle itself, the sine of the angle FGB, which is the complement of FGH to two right ones, may be considered as the sine of the angle of incidence; for which sine the line FB, as the sides of a triangle have the same ratio to each other, that the sines of their opposite angles have, may be taken. Again, the angle FBC is the angle of refraction, or its equal, because alternate to it, BFG, to which BG being an opposite side, may be taken as the sine. But FB is to BG in a ratio compounded of FB to BA, and of BA to BG, for the ratio that any two quantities bear to each other, is compounded of the ratio which the first bears to any other, and of the ratio which that other bears to the second. Now FB is to BA, supposing BD to vanish, as FD to DA; and BA is to BG, because of the parallel lines CB and FG, as AC to CF. That is, the ratio compounded of FD, the focal distance, to DA, the distance of the radiant point, and of AC, the distance between the radiant point and the center, to CF, the distance between the center and the focus, is equal to that which the sine of the angle of incidence bears to the sine of the angle of refraction.\*

Plate 6.  
Fig. 19.

\* Since the focal distance of the oblique ray AB is such that the compound ratio of FB to BA and of AC to CF shall be the same, whatever be the distance between B and D; it is evident, that, AC being always of the same length, the more the line AB lengthens, the more FB must lengthen too, or else FC must shorten; but if BF lengthens, CF will do so too, and in a greater ratio with respect to its own length than BF will, therefore the lengthening of BF will conduce nothing towards preserving the equality of the ratio: but as AB lengthens, BF and CF must both shorten, which is the only possible way in which the ratio can be continued the same. And it is also apparent, that the farther B moves from D towards O, the faster AB lengthens, and therefore the farther the rays enter from D, the nearer to the refracting surface is the place where they meet, but the space they are collected in, is the more diffused: and therefore in this case, as well as those taken notice of in the two preceding notes, different rays, though flowing from the same point, will constitute different foci; and none are so effectual as those which enter at or very near the point D. And since the same is observable of converging as well as of diverging rays, none, except those which enter very near that point, are usually taken into consideration; upon which account it is, that the distance DB, in determining the focal distances of diverging or converging rays entering a convex or concave surface, is supposed to vanish.

Case



Case 2. Of converging rays passing out of a rarer into a denser medium through a concave surface of the denser with such a degree of convergency, that they shall diverge after refraction.

Plate 6.  
Fig. 19.

Let the incident rays be HB and FD passing out of a rarer into a denser medium through the concave surface BD, and tending towards the point A, from whence the diverging rays flowed in the other case; then the oblique ray HB having its angle of incidence HBC equal to ABK the angle of incidence in the former case, will be refracted into the line BL such, that its angle of refraction KBL will be equal to FBC the angle of refraction in the former case; that is, it will proceed after refraction in the line FB produced, having the same focal distance FD with the diverging rays AB, AD, in the other case. But, by what has been already demonstrated, the ratio compounded of FD, the focal distance, to DA, in this case, the distance of the imaginary focus of the incident rays, and of AC, the distance between the same imaginary focus and the center, to CF, the distance between the center and the focus, is equal to that which the sine of the angle of incidence bears to the sine of the angle of refraction.

Case 3. Of diverging rays passing out of a rarer into a denser medium through a convex surface of the denser, with such a degree of divergency as to continue diverging.

Plate 6.  
Fig. 18.

Let AB, AD, be the diverging rays, and let their divergency be so great, that the refracted ray BL shall also diverge from the other; produce LB back to F, which will be the focal point; draw the radius CB, and produce it to K, produce BA likewise towards G, and draw FG parallel to BC. Then will ABK be the angle of incidence, whose sine BF may be taken for, as being opposite to the angle BGF, which is the complement of the other to two right ones. And LBC is the angle of refraction, or its equal KBF, or which is equal to this, BFG, as being alternate; therefore BG, the opposite side to this, may be taken for the sine of the angle of refraction. But BF is to BG, for the like reason as was given in Case the first, in a ratio compounded of BF to BA, and of BA to BG. Now BF is to BA, (DB vanishing) as DF to DA, and because of the parallel lines FG and BC, the triangles CBA and AGF are similar, therefore BA is to AG, as CA to AF; consequently, BA is to BA together with AG, that is, to BG, as CA is to CA together with AF, that is, CF. Therefore the ratio compounded of DF the focal distance to DA the distance of the radiant point, and of CA the distance between the radiant point and the center, to CF, the distance between the center and the focus, is equal to that which the sine of the angle of incidence bears to the sine of the angle of refraction.

SCHOL. By making HB and CD the incident rays, the proposition may be proved of converging rays passing out of a rarer into a denser medium, through a concave surface of the denser, with such convergency, that they shall continue to converge. Also, by the same method of reasoning as in the preceding cases, the proposition may be proved, of diverging rays passing out of a denser into a rarer medium through a concave surface of the denser, and of converging rays passing out of a denser into a rarer medium through a convex



a convex surface of the denser. And all those cases which are the converse of the preceding, admit of a similar proof; for, when rays pass out of one medium into another, the sine of the angle of incidence has the same ratio to the sine of the angle of refraction, as the sine of the angle of refraction has to the sine of the angle of incidence, when they pass through the same lines of direction the contrary way.

Case 4. Of rays passing out of a denser into a rarer medium, from a point between the center of convexity and the surface.

Let AB, AD, be two rays passing out of a denser into a rarer medium from the point A, which is taken between C the center of convexity and the refracting surface BD; through B draw CK, and let BL be the refracted ray; produce BL back to F, and draw FG parallel to BC. Then will ABC be the angle of incidence, of which BF, being opposite to its alternate and equal angle BGF, is the sine. LBK will be the angle of refraction, or its equal FBC, of which BG, being opposite to its complement to two right angles BFG, is the sine. But, BF is to BG in the compound ratio of BF to BA, and of BA to BG; and (BD vanishing) BF is to BA as DF to DA, and because the lines CB and FG are parallel, BA is to BG as CA to CF.

Plate 6.  
Fig. 20.

Case 5. Of rays passing out of a rarer into a denser medium from a point between the center of convexity and the surface.

Let AB, AD, be two diverging rays passing out of a rarer into a denser medium through the refracting surface BD, whose center of convexity is C, a point beyond that from whence the rays flow. Through B draw CK, and let BL be the refracted ray, produce it back to F, and draw FG parallel to BC meeting BA in G. ABC will be the angle of incidence, of which BF, being opposite to its complement to two right angles BGF, is the sine. The angle of refraction is LBK, or its equal FBC, of which BG, being opposite to its alternate and equal angle BFG, is the sine. But BF is to BG in the compound ratio of BF to BA and of BA to BG; and (BD vanishing) BF is to BA as DF to DA; and because of the parallel lines CB and GF, the triangles AFG and ABC are similar. BA therefore is to AG, as CA to AF; consequently, BA is to BA-AG, that is, to BG, as CA is to CA-AF, that is, to CF.

Plate 6.  
Fig. 21.

In like manner the Proposition may be proved of rays passing out of a denser into a rarer medium towards a point between the center of convexity and the surface, and in all other supposable cases.

COR. Hence the distance of the radiant point, or of the imaginary focus, being given, and also the radius of convexity, and the sines of the angles of incidence and refraction of diverging or converging rays, the focus of any lens may be thus found.

Let it be required to determine the focal distance of diverging rays passing out of air into glass through a convex surface, and let the distance of the radiant point be 20, and the radius of convexity be 5: let the focal distance be expressed by  $x$ : then, because by the



the preceding Proposition the ratio compounded of that which the focal distance bears to the distance of the radiant point (that is, in this supposition, of  $x$  to 20;) and of the ratio which the distance of the same radiant point from the center bears to the distance between the center and the focus (in this case, of 25 to  $x-5$ ) is equal to the ratio which the sine of the angle of incidence bears to the sine of the angle of refraction (that is, of 17 to 11,) we shall have in the instance before us, the following proportion,  $x : 20$   
 $25 : x-5 \} :: 17 : 11$ , and compounding them into one, which is done by multiplying the two first parts together, we have  $25x : 20x-100 :: 17 : 11$ , and multiplying the extream terms and middle terms together,  $340x-1700=275x$ , which equation after due reduction gives  $x=\frac{1700}{65}$ .

## S E C T. II.

*Of IMAGES produced by REFRACTION.*

## P R O P. XXIV.

Rays of light flowing from the several points of any object, farther from a convex lens than its principal focus, by passing through the lens, will be made to converge to points corresponding to those from which they proceeded, and will form an image.

Plate 6.  
Fig. 22.

Let ABC be a luminous or illuminated object. From every point, as A, B, C, rays diverge in all directions. Let some of these rays fall upon a convex lens GHK placed in a hole GK, in the window shutter of a dark room ML, at a greater distance from the object than the principal focal distance of the lens. BH being the axis, will (by Prop. XIV.) pass through the lens without refraction in the direction BHE. But the collateral rays BG, BK, made equally convergent by the lens, will cross the axis at E; that is, all the rays which come from the point B in the object, will be united behind the lens in the focus E. In like manner, among the rays AG, AH, AK, which diverge from the point A, whilst AH the axis (as was shewn Prop. XVII. Schol.) may be considered as if it went straight through the lens, the other rays will be made to converge, and will be united in a focus at F: and also, the rays from C will be united in D. The same may be shewn concerning every other point in the object. Consequently, there will be as many correspondent foci in the image as there are radiant points in the object: and these foci



foci will be disposed in the same manner with respect to one another as the radiants, and will therefore form an image. The object must be farther from the lens than its principal focus, else the rays from the several radiants would not converge, but either become parallel or diverging, (by Prop. XVII.) whence no image would be formed.

Exp. 1. Let the rays of the sun pass through a convex lens into a dark room, and fall upon a sheet of white paper placed at the distance of the principal focus from the lens.

2. The rays of a candle, in a room from which all external light is excluded, passing through a convex lens, will form an image on white paper.

## P R O P. XXV.

The image produced by rays of light passing through a convex lens is inverted.

The focus in which the rays that come from any point A, or B, are united, is, in the axis AHF, or BHE, of the beam, whether it fall directly or obliquely upon the lens. But the axis (by Prop. XVII.) is the middle ray of a cone of rays whose base is the surface of the lens, and vertex the radiant point. Every axis, therefore, as AHF, BHE, must pass through H the middle point of the lens, and consequently must cross one another in that point: from which it is manifest, that the rays from the lowest point C of the object will become the highest point of the image D; and that the image will be, with respect to the original object, inverted.

Plate 6.  
Fig. 22.

## P R O P. XXVI.

The image will not be distinct, unless the plane surface, on which it is received, be placed at the distance of the principal focus of the lens.

For otherwise the rays which come from a single point in the object, will not have its corresponding point in the image, but will be spread over a larger surface.

## P R O P. XXVII.

As an object approaches a convex lens, its image departs from it, and as the object departs the image approaches.

As the object ABC approaches the lens, the several radiants approach it: and consequently (by Prop. XIX.) the several foci which form the image FED recede; and the reverse. But the image can never be nearer the lens than its focus of parallel rays, since this is the place of the image, when the object is infinitely distant.

Plate 6.  
Fig. 22.



## P R O P. XXVIII.

When the object is placed parallel to the image, the diameter of the object is to the diameter of the distinct image, as the distance of the object from the lens, is to the distance of the image from the lens.

Plate 6.  
Fig. 22.

The radiant A (as appears from Prop. XXV.) is represented by its focus in the point F where the line AH, produced behind the lens, cuts a plane passing through the focus of parallel rays parallel to the lens. In like manner the radiant C is represented by its corresponding focus in the point D, in which the same plane is cut by the line CH produced. If therefore the distance of the extremities of the object, or its length, be AC, the length of its image will be DF. Since therefore AC is parallel to DF, the alternate angles ACD, CDF, (El. I. 29.) are equal; and also the alternate angles CAF, AFD: whence (El. VI. 4.) AC is to FD, as CH to DH, that is, the height of the object is to that of the image, as the distance of the object from the lens, to the distance of the image from the lens. Any diameter or line drawn across the object may be proved, in like manner, to have the same ratio to any corresponding diameter or line drawn across the image.

## P R O P. XXIX.

When the image appears confused, it is larger than when it is distinct.

For the rays, in this case, are not received upon the white surface exactly at the distance from the lens at which they are brought to a focal point, but either at a distance greater or less: and in either case the rays which come from any radiant points at the extremities A and C, will not be collected into points on the plane at F and D, but be spread over a small circular space round these points: whence the confused image will be larger than the distinct image.

## P R O P. XXX.

The object and the distinct image are similar surfaces.

Though the side of any object which is towards the lens be not a plane surface, yet the light is reflected from it in the same manner as if the figure of the object were drawn upon the plane surface of a piece of canvas, and differently shaded. Therefore the side of the object next to the lens may be considered as a plane figure. And since (by Prop. XXVIII.) the height of the object is to that of the picture, as the distance of the object from the lens, to the distance of the image from the lens, and also the breadth of the object

in



in any part, to the breadth of the image in the corresponding part in the ratio of these distances; it follows (El. V. 11.) that the height of the object is to the height of the image, as the breadth of the object in any part is to the breadth of the image in its corresponding part; that is, the object and image are similar surfaces.

## P R O P. XXXI.

The diameter of an image formed by rays passing from a given object through a convex lens, increases as the object approaches the lens, and decreases as the object recedes from the lens.

The diameter of the image (by Prop. XXVIII.) increases as its distance increases, and decreases as its distance decreases. And (by Prop. XXVII.) the distance of the image increases as the distance of the object decreases, and the reverse. Whence the diameter of the image increases as the distance of the object decreases, and decreases as the distance of the object increases.

## P R O P. XXXII.

When the distance of the object is given, the diameter of the image is as the diameter of the object.

If the object AC remains at the distance BH from the lens, the image (by Prop. XXVII.) will remain at the distance EH: whence the ratio of BH to EH, and consequently (by Prop. XXVIII.) the ratio of the diameter AC to its correspondent diameter DF, is given, or is invariable. Consequently, if AC increases or decreases, DF must proportionally increase or decrease, that is, the diameter of the image is directly as the diameter of the object.

Plate 6.  
Fig. 22.

## P R O P. XXXIII.

When the diameter and distance of the object are given, the diameter of the image will be as its distance from the lens.

If the diameter and distance of the object are given, it is manifest that the diameter of the image cannot be varied without changing the lens. But if, instead of the lens GHK, one less convex, or more convex, be used, the rays will be brought to a focus, and the image (by Prop. XXIV.) will be formed at a greater or less distance from the lens. And since (by Prop. XXVIII.) the distance of the object is always to the distance of the image, as the diameter of the object to the diameter of the image; the first and third terms remaining invariable, the second and fourth must be increased or diminished proportionally, that is, the diameter of the image will be directly as its distance from the lens.



## P R O P. XXXIV.

When the diameter and distance of the object are given, the area of the image is as the square of its distance from the lens.

Because the surface of the image (by Prop. XXX.) is similar to the surface of the object, whilst the surface of the object remains the same, the image, in every variation of its magnitude, must be similar to itself. But the areas of similar surfaces (El. VI. 20. Cor. 1.) are as the squares of their homologous diameters, that is, as the squares of their heights or breadths. Therefore the area of the image is always as the square of its diameter. And the diameter of the image, when the diameter and distance of the object are given, is (by Prop. XXXIII.) as its distance from the lens: therefore the square of its diameter, or its area, is as the square of its distance from the lens.

## P R O P. XXXV.

Though the distance of the object from the lens be varied, the image may be preserved distinct without varying the distance of the plane surface which receives it.

This will be the case, if as much as the image is brought forwards by the removal of the object, it is thrown backwards by diminishing the convexity of the lens, and the reverse: or the image may be preserved distinct without changing the lens, by increasing or diminishing the distance of the lens from the plane surface which receives the image, in the same ratio as the distance of the object from the lens is increased or diminished; which may be done either by moving the lens or the plane surface.

## P R O P. XXXVI.

The distances of the object and image, and the diameter of the object being given, the diameter of the image will not be altered by changing the area of the lens.

Plate 6.  
Fig. 22.

The height of the image DF is the distance between the two extreme foci F and D; the former of which is always in the axis AHF of the cone which has A for its vertex, and the latter in the axis CHD of the cone whose vertex is C, which axes cross each other in H. Since therefore DF, the height of the image, is the distance between these two lines AHF, CHD, where they meet the plane surface, the height of the image will be the same, whether the whole area GHK is open, or only a small part of it at H.

P R O P.



## P R O P. XXXVII.

When the object is near the lens, but not so near as the principal focus, in order to make the image distinct, the area of the lens must be small.

If the object was as near to the lens as the principal focus, or nearer, no image (by Prop. XVII.) could be formed. But let the object A be at a distance from the lens NP, very little greater than that of the principal focus: then the extreme rays AN, AP, of the cone NAP, diverging more than the rays AD, AE, if the plane surface which is to receive the rays, is placed where these rays are collected to a focus, the extreme rays AN, AP, diverging more, will not be collected, and the image on the plane surface will be confused. To prevent this, the extreme rays must be excluded by diminishing the area of the lens, or of the hole where it is placed. If the radiant A were at a greater distance, this would be unnecessary. Supposing the lens at SR, the extreme rays AN, AP, would pass above or below the lens, and only the middle rays, which are brought to a focus on the plane surface, would pass through the lens.

Plate 6.  
Fig. 23.

## P R O P. XXXVIII.

The brightness of an image, when its distance from the lens is given, is as the area of the lens.

When the whole area GHK is open, the entire cone of rays AGK passes through the lens from the point A, and is brought to a focus at F: but when the area is diminished to a small surface at H, the greatest part of the cone is excluded, and no rays, but the axis AH and those which are near it, can pass through the lens: whence it is manifest, that the focal point F must be more illuminated by the rays from A when the area of the lens is GHK, than when the area is diminished. The same may be said of every other cone of rays, and of every other point in the image. Therefore the whole image, although (by Prop. XXXVII.) made more distinct by diminishing the area, will be made fainter or less bright.

Plate 6.  
Fig. 22.

## P R O P. XXXIX.

The brightness of the image, when the area of the lens and the distance of the object are given, is inversely as the square of its distance from the lens.

The



The area of the lens and distance of the object being given, the number of rays which pass through the lens and form the image, is given. Now the same number of rays spread over a larger surface will not illuminate it so strongly as they would a smaller surface: that is, the brightness will be inversely as the illuminated area: and the area of the image is (by Prop. XXXIV.) as the square of its distance from the lens: whence its brightness is inversely as the square of its distance.

## P R O P. XL.

The heat at the focus of a burning glass, when the area of the glass is given, is inversely as the square of the focal distance.

The distance of the burning spot, that is the image of the sun, from the lens, is the focal distance, because the sun's rays are parallel. And because the heat and the brightness at the focus are as the number of rays collected, the heat is as the brightness. But the brightness (by Prop. XXXIX.) is inversely as the square of the distance of the image from the lens: therefore the heat is in the same ratio, that is, in this case, inversely as the square of the focal distance of the glass.

## P R O P. XLI.

The heat at the focus of a burning glass, when the focal distance is given, is as the area of the glass.

The brightness is (by Prop. XXXVIII.) as the area of the lens, and the heat is as the brightness: therefore the heat is also as the area of the lens.

## P R O P. XLII.

The heat at the focus of a burning glass is to the common heat of the sun, inversely as the area of the focus to the area of the glass.

The brightness, or the heat, must be inversely as the space or area over which the rays which cause them are spread, that is inversely as the area of the focus to the area of the glass.

SCHOL. This Proposition supposes all the rays which fall upon the lens to pass through it to the focus.

EXP.



EXP. Most of the preceding Propositions, from Prop. XIX. to XXXIX. may be confirmed, in a room from which all external light is excluded, by placing a convex lens, fixed in a frame which moves perpendicularly upon an oblong bar of wood, at distances, such as the Propositions require, from a lighted candle placed perpendicularly on the same bar of wood, and receiving the images upon white paper. Upon this bar of wood, on one side of a line over which the convex lens is placed, let a line perpendicular to the last mentioned line be divided into parts 1, 2, 3, 4, &c. each equal to the distance of the focus of parallel rays; and on the other side of the lens, let a line be divided in the same manner, and let the first division which is farther from the lens than the focus be subdivided into parts respectively equal to  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ , &c. of the distance of the focus of parallel rays: if a candle be placed over the division 2, it will form a distinct image on a paper held over the division  $\frac{1}{2}$ ; if the candle be over 3, the image will be at  $\frac{1}{3}$ , &c. whence it appears, that the distances of the correspondent foci vary reciprocally. Prop. XL, XLI, XLII, may be confirmed by holding a large double convex lens, or burning glass, in the sun's rays, and receiving the image on white paper, or other substances.



## C H A P. III.

## O F R E F L E C T I O N.

## S E C T. I.

*Of the LAWS of REFLECTION.*

DEF. XIX. A *Ray of Light*, turned back into the same medium in which it moved before its return, is said to be *reflected*.

DEF. XX. The *Angle of Reflection*, is that which is contained between the line described by a reflected ray and a line perpendicular to the reflecting surface at the point of reflection.

Plate 6.  
Fig. 25.

Let AC be the incident ray falling upon the reflecting surface <sup>B</sup>DE, CB will be the reflected ray, OC the perpendicular, ACO the angle of incidence, and OCB the angle of reflection.

## P R O P. XLIII.

The reflection of light from transparent bodies is either partial or total: the partial reflection happens either at the first or second surface, the total, at the second surface only.

When a beam of light falls upon a thick piece of polished glass, all the rays will not pass through it; but some of them will be reflected from the first surface of the glass, where the beam enters. At the second surface, some of the rays will also be reflected. These partial reflections happen, whatever is the obliquity of the rays. The total reflection happens when the angle of incidence, or the obliquity, is greater than 41 degrees. All the light which then comes to the second surface will be reflected.

SCHOL.



SCHOL. The rays of light are not reflected by striking upon the solid parts of bodies.

At least as many rays are reflected from the second surface when the light passes out of glass into air, as from the first when it passes out of air into glass: but if the reflection were caused by the striking of the rays upon solid parts of bodies, since glass is denser than air, that is, has more solid parts in a given space, a greater quantity of rays would be reflected from the first surface than from the second. Besides, it seems improbable that, at the second surface, with one degree of obliquity, the rays should meet with nothing but pores or interstices in the air, and pass freely into it, and that with a greater degree of obliquity, it should meet with nothing but solid parts, and be totally reflected. Again, since water is denser than air, if the reflection were owing to the striking of the rays upon the solid body, it might be expected that the light would be more perfectly reflected in passing out of glass into water than into air; whereas it is found, that if water be placed behind the glass instead of air, rays will not be reflected at the second surface, though their obliquity be greater than 41 degrees: from hence also it is manifest, that the reflection is not owing to the striking of the rays upon the second surface of the glass; for then it would be the same whatever were the medium beyond it.

#### P R O P. XLIV.

Reflection is caused by the powers of attraction and repulsion in the reflecting bodies.

Supposing that bodies attract those rays which are very near them, and repel those a little farther from them; and calling the space contiguous to the surface of the glass, where the rays are attracted, the attracting surface; and the space next to this, the repelling surface, the proposition may be thus proved.

Plate 6.  
Fig. 24.

Let GG, MM, be the repelling surface of a piece of glass, and Ra a ray falling upon it. This ray when it enters the surface will be retarded by the repulsion, and consequently refracted from the perpendicular at *a*. And this repulsion increasing till the ray gets into the surface of attraction, the ray will be constantly retarded, that is, turned out of its straight course at *b*, *c*, *d*, &c. till it become parallel to MM at *f* the limit of the repelling surface. And in this situation of the ray, the repelling force, which had retarded, will now constantly accelerate it, and consequently it will be continually refracted towards the perpendicular, at *g*, *h*, *i*, &c. till it emerge from the surface at *l*; when, the repelling force ceasing to act, the ray will proceed in a right line. Thus the ray, by reflection, is made to describe the curve *afl*.



## P R O P. XLV.

The angle of reflection is equal to the angle of incidence, and also their complements.

In all cases of reflection, the rays (by Prop. XLIV.) describe such a curve at  $afl$ . And since they describe one half of this curve by being retarded, and one half by being similarly accelerated, one half will be similar to the other; whence one half will make the same angle with a perpendicular at  $f$  as the other half makes with it. And the bending of the rays is made within so very small a compass, that is, the curve  $afl$  is so small that it may be neglected, as in fig. 25. where the angle  $ACO$  is equal to the angle  $BCO$ , and consequently, the angle  $ACD$  equal to  $BCD$ .

EXP. 1. Having described a semicircle on a smooth board, and from the circumference let fall a perpendicular bisecting the diameter, on each side of the perpendicular cut off equal parts of the circumference; draw lines from the points in which those equal parts are cut off to the center; place three pins perpendicular to the board, one at each point of section in the circumference, and one at the center; and place the board perpendicular to a plane mirror. Then look along one of the pins in the circumference to that in the center, and the other pin in the circumference will appear in the same line produced with the first, which shews that the ray which comes from the second pin, is reflected from the mirror at the center to the eye, in the same angle in which it fell on the mirror.

2. Let a ray of light passing through a small hole into a dark room, be reflected from a plane mirror; at equal distances from the point of reflection, the incident, and the reflected ray, will be at the same height from the surface.

## P R O P. XLVI.

All reflection is reciprocal.

Plate 6.  
Fig. 25.

If the ray  $AC$ , after it has been reflected in the line  $CB$ , is turned back again in the direction  $CB$ , it will be reflected (by Prop. XLV.) into  $AC$ . Therefore if  $ACO$  is the angle of incidence,  $OCB$  will be the angle of reflection; and if  $OCB$  be the angle of incidence,  $ACO$  will be the angle of reflection.

SCHOL. Sir Isaac Newton explains the cause of reflection by supposing, that light, in its passage from the luminous body, is disposed to be alternately reflected by, and transmitted through any refracting surface it may meet with; that these dispositions, which he calls Fits of easy reflection and easy transmission, return successively at equal intervals: and that they are communicated to it, at its first emission out of the luminous body it proceeds



ceeds from, probably by some very subtle and elastic substance diffused through the universe, in the following manner. As bodies falling into water, or passing through the air, cause undulations in each, so the rays of light may excite vibrations in this elastic substance. The quickness of these vibrations depending on the elasticity of the medium (as the quickness of the vibrations in the air, which propagate sound, depend solely on the elasticity of the air, and not upon the quickness of those in the sounding body) the motion of the particles of it may be quicker than that of the rays; and therefore when a ray, at the instant it impinges upon any surface, is in that part of a vibration of this elastic substance which conspires with its motion, it may be easily transmitted, and when it is in that part of a vibration which is contrary to its motion, it may be reflected. He farther supposes, that when light falls upon the first surface of a body, none is reflected there, but all that happens to it there is, that every ray that is not in a fit of easy transmission, is there put into one, so that when they come at the other side (for this elastic substance easily pervading the pores of bodies, is capable of the same vibrations within the body as without it) the rays of one kind shall be in a fit of easy transmission, and those of another in a fit of easy reflection, according to the thickness of the body, the intervals of the fits being different in rays of a different kind.

## P R O P. XLVII.

Rays of light reflected from a plane surface, have the same degree of inclination to each other that their respective incident ones have.

The angles of reflection of the rays AC, AI, AK, being equal to that of their respective incident ones, it is evident that each reflected ray will have the same degree of inclination to the surface DE, from whence it is reflected, that its incident one has; but it is here supposed that all those portions of surface, from whence the rays are reflected, are situated in the same plane; consequently, the reflected rays FC, LI, MK, will have the same degree of inclination to each other that their incident ones have, from whatever part of the surface they are reflected.

Plate 6.  
Fig. 26.

## P R O P. XLVIII.

Parallel rays reflected from a concave surface are made converging.

Let AF, CD, EB, represent three parallel rays falling upon the concave surface FB, whose center is C. To the points F and B draw the lines CF, CB; these being drawn from the center will be perpendicular to the surface at those points. The incident ray CD also passing through the center will be perpendicular to the surface, and therefore will return after reflection in the same line; but the oblique rays AF and EB will be reflected into the lines FM, BM, situated on the contrary side of their respective perpen-

Plate 6.  
Fig. 27.



diculars CF and CB. They will therefore proceed converging after reflection towards some point, as M, in the line CD.

## P R O P. XLIX.

Converging rays falling upon a concave surface are made to converge more.

Plate 6.  
Fig. 27.

Let GB, HF, be the incident rays. Now, because these rays have larger angles of incidence than the parallel ones, AF, EB, in the foregoing case, their angles of reflection will also be larger than theirs: they will therefore converge after reflection, suppose in the lines FN, and BN, having their point of concurrence N farther from C than the point M, to which the parallel rays AF and EB converged in the foregoing case.

## P R O P. L.

Diverging rays, falling upon a concave surface, if they diverge from the focus of parallel rays, become parallel; if from a point nearer the surface than that focus, will diverge less than before reflection; if from a point between that focus and the center, will converge after reflection to some point, on the contrary side of the center, and farther from the center than the point from which it diverged; if from a point beyond the center, the reflected rays will converge to a point on the contrary side, but nearer to it than the point from which they diverged; if from the center, they will be reflected thither again.

Plate 6.  
Fig. 27.

Let the incident diverging rays be MF, MB, proceeding from M the focus of parallel rays; then as the parallel rays AF and EB were reflected into the lines FM and BM, these rays will now on the contrary be reflected into them.

Let them diverge from N, a point nearer to the surface than the focus of parallel rays, they will then be reflected into the diverging lines HF and BG, which the incident rays GB and HF described, which were shewn to be reflected into them in the foregoing proposition; but the degree wherein they diverge, will be less than that wherein they diverged before reflection.

Let them proceed diverging from X, a point between the focus of parallel rays and the center, they then make less angles of incidence than the rays MF and MB, which became



became parallel by reflection, they will consequently have less angles of reflection, and proceed therefore converging towards some point, as Y; which point will always fall on the contrary side the center, because a reflected ray always falls on the contrary side the perpendicular with respect to that on which its incident one falls; and therefore will be farther distant from the center than X.

If the incident ones diverge from Y, they will after reflection converge to X, those which were the incident rays in the former case being the reflected ones in this.

Lastly, if the incident rays proceed from the center, they fall in with their respective perpendiculars, and for that reason are reflected thither again.

EXP. Place a concave mirror at proper distances from an open orifice, or a convex, or a concave lens, through which a beam of solar rays passes, according to the three preceding propositions.

## P R O P. LI.

Parallel rays reflected from a convex surface are made diverging.

Let AB, GD, EF, be three parallel rays falling upon the convex surface BF, whose center of convexity is C, and let one of them, GD, be perpendicular to the surface. Through B, D, and F, the points of reflection, draw the lines CV, CG, and CT, which because they pass through the center will be perpendicular to the surface at those points. The incident ray GD, being perpendicular to the surface, will return after reflection in the same line, but the oblique ones AB and EF in the lines BK and FL situated on the contrary side their respective perpendiculars BV and FT. They will therefore diverge after reflection, as from some point M in the line GD produced.

Plate 6.  
Fig. 28.

## P R O P. LII.

Diverging rays reflected from a convex surface are made more diverging.

Let GB, GF, be the incident rays. These having larger angles of incidence than the parallel ones AB and EF in the preceding case, their angles of reflection will also be larger than theirs; they will therefore diverge after reflection, suppose in the lines BP and FO, as from some point N farther from C than the point M; and the degree wherein they will diverge, will exceed that wherein they diverged before reflection.

Plate 6.  
Fig. 28.

PROP.



## P R O P. LIII.

Converging rays reflected from a convex surface, if they tend towards the focus of parallel rays, will become parallel; if to a point nearer the surface than that focus, will converge less than before reflection; if to a point between that focus and the center, will diverge as from a point on the contrary side of the center farther from it than the point towards which they converged; if to a point beyond the center, they will diverge as from a point on the contrary side of the center nearer to it than the point towards which they first converged; and if towards the center, they will proceed, on reflection, as from the center.

Plate 6.  
Fig. 28.

Let the converging rays be KB, LF, tending towards M the focus of parallel rays; then, as the parallel rays AB, EF, were reflected into the lines BK and FL, those rays will now on the contrary be reflected into them.

Let them converge in the lines PB, OF, tending towards N, a point nearer the surface than the focus of parallel rays, they will then be reflected into the converging lines BG and FG, in which the rays GB, GF, proceeded, which were shewn to be reflected into them in the last proposition immediately foregoing; but the degree wherein they will converge, will be less than that wherein they converged before reflection.

Let them converge in the lines RB and SF proceeding towards X, a point between the focus of parallel rays and the center; their angles of incidence will then be less than those of the rays KB and LF, which became parallel after reflection, their angles of reflection will therefore be less, on which account they must necessarily diverge, suppose in the lines BH and FI, from some point, as Y; which point will fall on the contrary side the center with respect to X, and will be farther from it than X.

If the incident rays tend towards Y, the reflected ones will diverge as from X, those which were the incident ones in one case, being the reflected ones in the other.

And lastly, if the incident rays converge towards the center, they fall in with their respective perpendiculars; on which account they proceed after reflection, as from thence.

Exp. Illustrate the three preceding propositions by receiving upon a convex mirror, a solar ray passing through an open orifice, or a concave or convex lens.

P R O P.



## P R O P. LIV.

When rays fall upon a plane surface, if they diverge, the focus of the reflected rays will be at the same distance behind the surface, that the radiant point is before it: if they converge, it will be at the same distance before the surface, that the imaginary focus of the incident rays is behind it.

Case 1. Of diverging rays. Let AB, AC, be two diverging rays incident in the plane surface DE, the one perpendicularly, the other obliquely; the perpendicular one AB will be reflected to A proceeding as from some point in the line AB produced; the oblique one AC will be reflected into some line CF, such that the point G, where the line FC produced intersects the line AB produced also, shall be at an equal distance from the surface DE with the radiant A. For the perpendicular CH being drawn, ACH and HCF will be the angles of incidence and reflection, which being equal, their complements ACB and FCE are so too: but the angle BCG is equal (El. I. 15.) to FCE: therefore in the triangles ABC and GBC the angles at C are equal, the side BC is common, and the angles at B are also equal to each other, as being right ones; therefore (El. I. 26.) the lines AB and BG, opposite to the equal angles at C, are also equal, and consequently the point G, the focus of the incident rays AB, AC, is at the same distance behind the surface, that the point A is before it.

Plate 6.  
Fig. 26.

Case 2. Of converging rays. Supposing FC and AB to be two converging incident rays, CA and BA will be the reflected ones (the angles of incidence in the former case being now the angles of reflection, and the reverse) having the point A for their focus; but this, from what was demonstrated above, is at an equal distance from the reflecting surface with the point G, which in this case is the imaginary focus of the incident rays FC and AB. What is here demonstrated of the ray AC, holds equally of any other, as AI, AK, &c.

SCHOL. The case of parallel rays incident on a plane surface, is included in this proposition; for in that case we are to suppose the radiant to be at an infinite distance from the surface, and then by the proposition, the focus of the reflected rays will be so too; that is, the rays will be parallel after reflection, as they were before.

## P R O P. LV.

When parallel rays are incident upon a spherical surface, the focus of the reflected rays will be the middle point between the center of convexity and the surface.

Case



Plate 7.  
Fig. 1.

Case 1. Of parallel rays falling upon a convex surface. Let AB, DH, represent two parallel rays incident on the convex surface BH, the one perpendicularly, the other obliquely; and let C be the center of convexity; suppose HE to be the reflected ray of the oblique incident one DH proceeding as from F, a point in the line AB produced. Through the point H draw the line CI, which will be perpendicular to the surface at that point, and the angles DHI and IHE, being the angles of incidence and reflection, will be equal. But HCF is equal (El. I. 29.) to DHI, and CHF (El. I. 15.) to IHE; wherefore the triangle CFH is isosceles; and consequently the sides CF and FH are equal: but supposing BH to vanish, FH is equal to FB, and therefore upon this supposition FC and FB are equal, that is, the focus of the reflected rays is the middle point between the center of convexity and the surface.

Plate 7.  
Fig. 2.

Case 2. Of parallel rays falling upon a concave surface. Let AB, DH be two parallel rays incident, the one perpendicularly, the other obliquely, on the concave surface BH, whose center of concavity is C. Let BF and HF be the reflected rays meeting each other in F; this will be the middle point between B and C. For drawing through C the perpendicular CH, the angles DHC and FHC, being the angles of incidence and reflection, will be equal, to the former of which the angle HCF is equal, as alternate; and therefore the triangle CFH is isosceles. Wherefore CF and FH are equal: but if we suppose BH to vanish, FB and FH are also equal, and therefore CF is equal to FB; that is, the focal distance of the reflected rays is the middle point between the centre and the surface.\*

SCHOL. The converse of these two cases may be demonstrated in a similar manner, by making the incident rays the reflected ones.

Plate 7.  
Fig. 3.

\* It is here observable, that the farther the line DH is taken from AB, the nearer the point F falls to the surface. For the farther the point H recedes from B, the larger the triangle CFH will become; and consequently since it is always an isosceles one, and the base CH, being the radius, is every where of the same length, the equal legs CF and FH will lengthen; but CF cannot grow longer unless the point F approach towards the surface. And the farther H is removed from B, the faster F approaches to it. This is the reason, that whenever parallel rays are considered, as reflected from a spherical surface, the distance of the oblique ray from the perpendicular ray is taken so small with respect to the focal distance of that surface, that, without any physical error, it may be supposed to vanish. From hence it follows, that if a number of parallel rays, as AB, CD, EG, &c. fall upon a convex surface, and if BA, DK, the reflected rays of the incident ones AB, CD, proceed as from the point F, those of the incident ones CD, EG, namely, DK, GL, will proceed as from N, those of the incident ones EG, HI, as from O, &c. because the farther the incident ones CD, EG, &c. are from AB, the nearer to the surface are the points F, f, f, in the line BF, from which they proceed after reflection; so that properly the foci of the reflected rays BA, DK, GL, &c. are not in the line AB produced, but in a curve line passing through the points F, N, O, &c.

The same is applicable to the case of parallel rays reflected from a concave surface, as expressed by the dotted lines on the other half of the figure, where PQ, RS, TV, are the incident rays; QE, Sf, Vf, the reflected ones intersecting each other in the points X, Y, and F; so that the foci of those rays are not in the line FB, but in a curve passing through those points.



## P R O P. LVI.

When rays fall upon any spherical surface, if they diverge, the distance of the focus of the reflected rays from the surface is to the distance of the radiant point from the same (or, if they converge, to that of the imaginary focus of the incident rays) as the distance of the focus of the reflected rays from the center is to the distance of the radiant point (or imaginary focus of the incident rays) from the same.

Case 1. Of diverging rays falling upon a convex surface. Let RB, RD, represent two diverging rays flowing from the point R as from a radiant, and falling the one perpendicularly, the other obliquely, on the convex surface BD, whose center is C. Let DE be the reflected ray of the incident one RD; produce ED to F, and through R draw the line RH parallel to FE, till it meets CD produced in H. Then will the angle RHD be equal (El. I. 29.) to EDH the angle of reflection, and therefore equal also to RDH, which is the angle of incidence; wherefore the triangle DRH is isosceles, and consequently DR is equal to RH. Now the lines FD and RH being parallel, (El. VI. 2.) FD is to RH, or its equal RD, as CF to CR; but BD vanishing, FD and RD differ not from FB and RB; wherefore FB is to RB also, as CF to CR; that is, the distance of the focus from the surface is to the distance of the radiant point from the same, as the distance of the focus from the center is to the distance of the radiant from thence.

Plate 7.  
Fig. 4.

Case 2. Of converging rays falling upon a concave surface. Let KD and CB be the converging incident rays, having their imaginary focus in the point R, which was the radiant in the foregoing case. Then as RD was in that case reflected into DE, KD will in this be reflected into DF; for, since the angles of incidence in both cases are (El. I. 15.) equal, the angles of reflection will be so too; so that F will be the focus of the reflected rays: but it was there demonstrated, that FB is to RB as CF to CR, that is, the distance of the focus from the surface, is to the distance (in this case) of the imaginary focus of the incident rays, as the distance of the focus from the center is to the distance of the imaginary focus of the incident rays from the same.

Plate 7.  
Fig. 4.

Case 3. Of converging rays falling upon a convex surface, and tending to a point between the focus of parallel rays and the center. Let BD represent a convex surface, whose center is C, and focus of parallel rays is P; and let AB, KD, be two converging rays incident upon it, and having their imaginary focus at R, a point between P and C. Now because KD tends to a point between the focus of parallel rays and the center, the reflected ray DE will diverge from some point on the other side the center, suppose F; as was shewn Prop. LIII. Through D draw the perpendicular CD, and produce it

Plate 7.  
Fig. 5.

Z

to



to H, then will KDH and HDE be the angles of incidence and reflection, which being equal, their vertical ones RDC and CDF will be so too, and therefore the vertex of the triangle RDF is bisected by the line DC: whence (El. VI. 3.) FD and DR, or, BD vanishing, FB and BR are to each other as FC to CR; that is, the distance of the focus of the reflected rays is to that of the imaginary focus of the incident ones, as the distance of the former from the center is to the distance of the latter from the same.

Plate 7.  
Fig. 5.

Case 4. Of diverging rays falling upon a concave surface, and proceeding from a point between the focus of parallel rays and the center. Let RB, RD, be the diverging rays incident upon the concave surface BD, having their radiant in the point R, the imaginary focus of the incident rays in the foregoing case. Then as KD was in that case reflected into DE, RD will now be reflected into DF. But it was there demonstrated that FB and RB are to each other as CF to CR; that is, the distance of the focus is to that of the radiant, as the distance of the former from the center is to the distance of the latter from the same.\*

SCHOL. 1. If the reflected ray be made the incident one, those cases which are respectively the converse of the foregoing may be demonstrated in the same manner.

SCHOL. 2. Let it be required to find the focal distance of diverging rays incident upon a convex surface, whose radius of convexity is 5 parts, and the distance of the radiant from the surface is 20.

Call the focal distance sought  $x$ , then will the distance of the focus from the center be  $5-x$ , and that of the radiant from the same 25; therefore (by Prop. LVI.) we have the following proportion,  $x : 20 :: 5-x : 25$ , and, multiplying extremes together and means together, we have  $25x = 100 - 20x$ , which after due reduction gives  $x = \frac{100}{45}$ .

If in any case it should happen, that the value of  $x$  should be a negative quantity, the focal point must then be taken on the contrary side of the surface.

Plate 7.  
Fig. 6.

\* In the case of diverging rays falling upon a convex surface, the farther the point D is taken from B, the nearer the point F, the focus of the reflected rays, approaches to B, while the radiant R remains the same. For it is evident from the curvature of a circle, that the point D may be taken so far from B, that the reflected ray DE shall proceed as from F, G, H, or even from B, or from any point between B and R, and the farther it is taken from B, the faster the point, from which it proceeds, approaches towards R. The like is applicable to any of the other cases of diverging or converging rays incident upon a spherical surface. This is the reason that, when rays are considered as reflected from a spherical surface, the distance of the oblique rays from the perpendicular one is taken so small, that it may be supposed to vanish. From hence it follows, that if a number of diverging rays are incident upon the convex surface BD at the several points B, D, D, &c. they shall not proceed after reflection as from any one point in the line RB produced, but as from a curve line passing through the several points F,  $f$ ,  $f$ , &c. The same is applicable in all the other cases.



## S E C T. II.

*Of IMAGES produced by REFLECTION.*

## P R O P. LVII.

When any point of an object is seen by reflected light, it appears in the direction of that line which the ray describes after its last reflection.

Since reflection gives the same direction to the rays as if they had originally come from the place from which the reflected rays diverge, an object seen by reflection must appear in that place. The visible image must therefore consist of imaginary radiants diverging from thence.

## P R O P. LVIII.

In all mirrors, plane or spherical, the place of the imaginary radiant, when it is determined, is the intersection of the perpendicular from the radiant to the mirror, and any reflected ray.

Let D be a radiant in any object DE, and DF a ray from this radiant reflected in the line FC; draw DI the perpendicular from the radiant to the mirror, and produce CF, DI, till they meet in L; this point will be the imaginary radiant. Because the ray DI falls perpendicularly upon the mirror, it will be reflected back in the same line ID, and therefore will appear to come from some point in DI produced. And since (from Prop. LVII.) all rays which diverge from the same real radiant before reflection, must diverge from the same imaginary radiant after reflection, any other ray from D, as FC, must appear to diverge from the same imaginary radiant with the ray DI, that is, from some point in DI: but the ray FC (by Prop. LVII.) appears after reflection to proceed in the line FC: it must therefore appear to come from some point in FC, and also from some point in DI, that is, from the point L, in which DI intersects FC. The imaginary radiant of the rays DI, DF, after reflection is therefore L, the intersection of the perpendicular and the reflected ray.

Plate 7.  
Fig. 7.

DEF. XXI. The passage of reflection is the incident ray added to the reflected ray: as  $DF + FC$ .



## P R O P. LIX.

In plane mirrors, the distance of the last image from the mirror is equal to the distance of the object from it, and the distance of any point in the last image from the eye is equal to the passage of reflection.

Plate 7.  
Fig. 7.

The distances of the imaginary radiants  $L$ ,  $M$ , behind the mirror are (by Prop. LIV.) respectively equal to the distances of the corresponding real radiants  $D$ ,  $E$ , before the mirror: therefore the distance of the last image  $L$ ,  $M$ , made up of imaginary radiants between  $L$  and  $M$ , corresponding to real radiants in the object  $DE$ , is equal to the distance of that object. The distance of  $L$ , the highest point of the image, from  $C$ , any given place of the eye, is  $CFL$ , equal to  $DFC$  the passage of reflection, because (by Prop. LIV.)  $LF$  is equal to  $DF$ . The same may be shewn of  $M$ , or any other point in the image.

## P R O P. LX.

In plane mirrors, the image is equal and similar to the object.

Plate 7.  
Fig. 7.

If  $D$  is the highest point of the object, the highest point of the image is (by Prop. LVII.) in the perpendicular  $DIL$ ; and if  $E$  be the lowest point of the object, the lowest point of the image is in the perpendicular  $EZM$ . But  $DIL$  and  $EZM$  are parallel (El. XI. 6.) because they are both perpendicular to the plane surface  $AB$ . Consequently, the distance between these lines, that is, the heights of the object and image,  $DE$ ,  $LM$ , are equal. In like manner it may be shewn, that any diameter of the object is equal to its corresponding diameter in the image: whence the object and image are in all respects equal, and consequently similar, surfaces.



## C H A P. IV.

## O F V I S I O N.

## S E C T. I.

*Of the LAWS of VISION.*

## P R O P. LXI.

When the rays which come from the several points in any object enter the eye, they will paint an inverted image upon the *retina*.

Let ABA be a section of an eye. AB, BA, is the *tunica sclerotica*, a white coat which encompasses the globe of the eye, except the fore-part between A and A. The fore-part AA is covered by a transparent coat, a little more protuberant than any other part of the eye, called the *tunica cornea*. In the cavity of the eye is placed a convex lens Ce, consisting of a hard transparent substance called the *chrystalline humour*. This humour is kept in its place and fixed to the coats by certain ligaments all round it at ee, called *ligamenta ciliaria*. Under the *tunica cornea*, and at some little distance from it, is the *iris* o, o, which has in it a small orifice, called the pupil of the eye. This iris is tinged with variety of colours, from which the eye is said to be blue, hazel, black, &c. It consists of muscular fibres, which can contract or dilate the pupil. The remaining part between the *cornea* and the *chrystalline humour* is filled with a thin transparent fluid, like water, called the *aqueous humour*. SeN, is a white coat which consists of the fibres of the optic nerve woven together like a net: this coat is called the *retina*. Between the *sclerotica* and the *retina* is another coat which is called the *choroides*. The cavity of the eye, between the *chrystalline humour* and the *retina*, is filled with a transparent substance, neither so fluid as the aqueous humour nor so hard as the *chrystalline*, called the *vitreous humour*. The *optic nerve* MBB is inserted at N.

Plate 7.  
Fig. 10.

Through the pupil the rays which diverge from the several points of any object ABC pass into the cavity. The rays first pass into the *cornea* IK, which is a meniscus like a watch glass, and therefore has no considerable effect upon the rays. The fore-part of the aqueous humour under the cornea is convex; and therefore, being denser than the air, it will make the several diverging beams from ABC (by Prop. XVII.) diverge less. The

Fig. 11.

rays



rays then pass through the pupil; and next, through the chrySTALLINE humour, which, being a convex lens, will make them (by Prop. XVII.) converge to as many points upon the retina LDF. Consequently at DEF, or somewhere upon the retina, as upon a piece of white paper in a dark room, an inverted image of the object (by Prop. XXV.) will be painted.

EXP. Take off the *sclerotica* from the back part of the eye of an ox, or other animal, and place the eye in the hole of the window shutter of a dark room, with its fore-part towards the external objects; a person in the room will, through the transparent coat, see the inverted image painted upon the *retina*.

Plate 7.  
Fig. 11.

DEF. XXII. The optic axis, is the axis of the chrySTALLINE humour continued to the object at which we look. The axis PO of the chrySTALLINE humour GPH, continued to the point B, is the optic axis directed towards that point.

DEF. XXIII. That point of the *retina*, upon which the optic axis continued back would fall, is called the middle of the *retina*. If OP be continued back to E, the point E is the middle of the *retina*.

#### P R O P. LXII.

The images upon the *retina* are the cause of vision.

It is found from experience, that when the image upon the *retina* is bright, the object is clearly seen; and when the image is faint, the object appears faint: also, that when the image is distinct, the object is seen distinctly; and when the image is confused, the object appears confused. Hence it may be concluded, that these images are the cause of vision.

#### P R O P. LXIII.

The point in any object towards which the optic axis is directed, is seen more distinctly than the rest.

It is known from experience, that when the eye is turned directly towards any part of an object, that is, when the optic axis is directed towards that point, though the whole object, if it be not very large, will be seen, that part on which we look directly will appear most distinct, and the other portions of the object being drawn on parts of the *retina*, somewhat nearer to the chrySTALLINE humour than the middle point of the *retina* will appear a little confused.

P R O P.



## P R O P. LXIV.

Objects appear erect, although their images on the *retina* are inverted.

This is known by experience, and is not inconsistent with the explanation above given of vision. For it is not the image in the *retina*, but the object itself, which we see, and we judge of its relative position, by moving the point of distinct vision along the object, and determine that part to be the highest which requires the eye to be the most lifted up in order to see it distinctly.

## P R O P. LXV.

An object may appear single, although it is seen by both eyes at once.

If both eyes are turned directly to the object C, that is, if the optic axes AC, AB, meet in the object, it will appear single. But if, whilst one eye is turned towards C, the corner of the other is pressed with the finger so as to alter the position of its optic axis, the object will then appear double. For when one eye is turned towards an object and the other turned a different way, the same object will be seen by each eye in a different direction, that is, one eye will see it in one place, and the other in another, from whence it must appear double. But, if both eyes are directed the same way, that is, to the place of the object, though two objects may be said to be seen, yet as both of them are alike, and seen in the same place, they will appear but as one. If whilst both eyes are directed towards C, another object D be placed at some considerable distance directly beyond it, the object D will appear double; for since the eyes see the object D without being turned directly towards it, the place of D is indeterminate; to the right eye it appears on the right hand of C, and to the left eye on the left of C, that is, being seen in two places, it must appear double. If the sight be directed to the farther object D, the nearer object C will appear double. For, the object C is seen by the right eye in the direction of a line which passes on the left side of D, and by the left eye in the direction of a line which passes on the right of D. In both cases, one of the objects appears double, when the eyes are not directly fixed upon it, that is, when the optic axes do not meet in it; and the other object appears single, when the eyes are both directly fixed upon it, that is, when the optic axes meet in the object.

Plate 7.  
Fig. 12.

EXP. I. View a nearer and a more distant object at the same time, according to the proposition.

2. Let a pasteboard, having a hole in it, be fixed between the eyes of a spectator and two candles, so placed, that when the right eye is shut, the left eye may see only  
one



one candle, and when the left eye is shut, the right eye can see only the other, although both candles are visible; if both eyes be fixed steadfastly upon the hole, they will appear as one candle placed at the hole.

## P R O P. LXVI.

There is one part of the *retina* where no perception of the object is conveyed to the mind by the image formed upon it.

This is found by Experiment. Place two small circles of white paper upon a dark coloured wall at the height of the eye, and at the distance of near two feet from each other. If the spectator, at a proper distance, shuts his right eye and looks with the left directly at the paper on his right hand, he will not see the left hand paper, although the objects round it are visible. Hence it is to be inferred, that the rays from the left hand paper fall upon a part of the *retina* which is insensible.

SCHOL. It is supposed that this part of the *retina* is that where the optic nerve is inserted: and because the coat called the choroides touches the *retina* in all other parts but is discontinued here, it has been conjectured that the seat of vision is not the optic nerve, but the choroides; but this point remains undetermined.

## P R O P. LXVII.

If the chrystalline humour has either too much or too little convexity, the sight will be defective.

In persons who are short-sighted, the humours of the eye are too convex, and bring the rays to a focus before they reach the retina, unless the object be brought near to it; in which case (by Prop. XXVII.) the image is cast farther back. In others, the humours of the eye have so little convexity, that the focal point lies behind the *retina*; whence, unless the object is removed to a great distance from the eye, the vision will be indistinct.

Plate 7.  
Fig. 11.

DEF. XXIV. The *optic angle* in viewing any object is the angle at the eye subtended by the diameter of the object. The angle AOC, subtended by AC the diameter of the object, is the optic angle.

## P R O P. LXVIII.

The apparent diameter of any object, is proportional to the diameter of the image of that object in the *retina*.



To an eye placed at O, the apparent magnitude of the object ABC is that visible extension which lies between A and C. If two lines AO, CO, are drawn from these points crossing one another at the eye, they will be the axes of the pencils which come from A and C, and will contain between them the diameter AC; and the points A, C, will be represented on the *retina* (by Prop. LXI.) at F and D: consequently, DF will be the diameter of the image; and this diameter is contained between the two lines AO, CO, produced to the retina. Now it is manifest, that the visible extension contained between AO and CO, that is the apparent diameter of the object, is as the angle AOC; and that the diameter of the image contained between DO and FO is as the angle DOF. But the angles DOF, AOC, (El. I. 15.) are equal. Therefore the apparent diameter of the object, and the diameter of the image, are each of them proportional to the same angle, and consequently proportional to one another.

Plate 7.  
Fig. 11.

SCHOL. When we speak of an optic angle, it is not meant that we see the point in which the optic axes meet; but, since by experience we learn to judge of such distances as are not very great, by the sensations accompanying the different inclinations of the eye, which are analogous to the optic angle, we express these different inclinations of the eye, by that angle. In like manner, although the eye does not see a pencil of rays, whilst the breadth of the pupil bears a sensible proportion to the distance of the focus from which the rays diverge to the eye, we have sensations from which experience enables us to judge of the place of that focus. So, the magnitude of an image upon the retina being always proportional to that of the visual angle of the object, though that angle is not actually measured by the eye, the difference of sensations accompanying different magnitudes of the image enable us to distinguish different visual angles. Thus it appears, that the use of lines and angles in optics, has its foundation in nature.

### P R O P. LXIX.

When the diameter of an object is given, its apparent diameter is inversely as its distance from the eye.

The apparent diameter of an object (Prop. LXVIII.) is as the diameter of its image upon the retina: and (Prop. XXXI.) the diameter of the image, when the object is given, is inversely as the distance of the object. Therefore the apparent diameter of the object is also inversely as the distance of the object. The same may be proved of any apparent length whatsoever.

COR. I. Hence the apparent diameter of an object may be magnified in any proportion: for the less the distance of the eye from the object, the greater will be its apparent diameter. But without the help of glasses, an object brought nearer the eye than about five inches, though it appears larger, will at the same time be seen confusedly.

A a

COR.



Plate 7.  
Fig. 15.

COR. 2. Hence parallel lines, as ABC, DEF, seen obliquely, appear to converge more and more, as they are farther extended from the eye : for the apparent magnitude of their perpendicular intervals, as AD, BE, CF, &c. are perpetually diminished.

P R O P. LXX.

The apparent diameter of an object, whose distance is given, is directly as its real diameter.

The apparent diameter of an object (by Prop. LXVIII.) is as the diameter of its image : and the diameter of the image (by Prop. XXXIII.) when the distance of the object is given, is as the diameter of the object. Therefore the apparent diameter of an object whose distance is given, is as its real diameter.

P R O P. LXXI.

The apparent diameters of different objects at different distances from the eye will be equal, if their real diameters are as their distances.

For (by Prop. XXXIV.) the diameters of their respective images upon the *retina* will be equal ; and their apparent diameters (by Prop. LXVIII.) are as the diameters of their images.

P R O P. LXXII.

The apparent length of an object seen obliquely, is as the apparent length of a subtense of the optic angle perpendicular to the optic axis.

Plate 7.  
Fig. 13.

If DF is an object seen obliquely, and DG an object seen directly, that is, if DF is oblique, and DG perpendicular to the optic axis OR, then supposing them to subtend the same angle DOF, their images upon the *retina* (Def. XXIV.) will be equal, whence (by Prop. LXVIII.) their apparent diameters will be equal. Consequently, the greater the subtense GD is, the greater will be the apparent length of the object DF; and the reverse.

COR. Hence an object appears shortened by being seen obliquely.

P R O P. LXXIII.

When equal objects are seen obliquely, their apparent lengths are inversely as the squares of their distances from the eye.

Let



Let the eye be at  $O$ ; and in the line  $BC$  at different distances from the eye take equal spaces  $DF, df$ . The apparent length of  $DF$  (by Prop. LXXII.) is proportional to the apparent length of  $GD$ , the subtense of the optic angle  $DOF$ , perpendicular to the optic axis  $OR$ . In like manner, the apparent length of  $df$  is proportional to that of  $gd$ , the subtense of  $dOf$ . Now  $GD, gd$ , are subtenses also of the angles  $GFD, gfd$ : and as the side  $Of$  is to the side  $OF$ , so is the sine of the angle  $OfF$ , that is, of its supplement  $OFB$ , to the sine of the angle  $OfB$ , or  $OfF$ . Hence, since small angles are to one another nearly as their sines, if these are small angles,  $Of$  will be to  $OF$  as the angle  $OFB$  to the angle  $OfB$ ; that is,  $Of$  will be to  $OF$ , as the subtense  $GD$  to the subtense  $gd$ , or  $GD$  is to  $gd$  inversely as  $OF$  to  $Of$ ; that is, the subtenses of the optic angles, and consequently, from what has been shewn, the apparent diameters  $DF, df$ , are inversely as their distances from the eye. This proportion arises from the different degrees of obliquity at which the eye sees the equal spaces  $DF, df$ . But, if their obliquities with respect to the eye were the same, the apparent length of  $DF$  (by Prop. LXIX.) would be to that of  $df$  inversely as their distances. Since then the apparent lengths of  $DF, df$ , are inversely as their distances on account of their different obliquities, and also inversely as their distances on account of their different distances; on both accounts taken together, they are in the ratio compounded of the inverse ratio of their distances, and the same, that is, inversely as the squares of their distances.

Platè 7.  
Fig. 13.

## P R O P. LXXIV.

The apparent diameter of an object is not changed by contracting or dilating the pupil.

For, when the distance is given, the diameter of the image (by Prop. XXXVI. Schol. 2.) remains the same whatever be the area of the pupil, and consequently (by Prop. XXXIII.) the apparent diameter of the object.

## P R O P. LXXV.

An object appears larger when it is seen confusedly, than when it is seen distinctly.

For the confused image (by Prop. XXIX.) is larger than the distinct image, and consequently (by Prop. LXVIII.) the apparent magnitude of the object is greater when it is seen confusedly than when it is seen distinctly.

COR. Hence objects appear magnified when seen through a mist; the drops of which refract the rays so differently, that they cannot be collected into one focus.



## P R O P. LXXVI.

A spectator in motion sees an object at rest, moving the contrary way.

Plate 7.  
Fig. 14.

If while an object at P is at rest, the eye be carried parallel to PQ in the direction from Q towards P, its image on the *retina* will move from *p* to *q*: the same effect will be produced, if the object moves from P towards Q; and if the velocity of the object and the eye, in each case, be the same, the apparent velocity will be the same also.

COR. An object moving along a line PK will appear at rest to a spectator moving along the line QG, parallel to PK: if the motion of the object be quicker or slower than that of the spectator, it will have an apparent motion direct or retrograde: if the two motions are in contrary directions, the apparent motion of the object will vary with the real motion of the spectator.

## P R O P. LXXVII.

The same degree of velocity appears greatest, when the motion is in a line perpendicular to the optic axis; and when the motion is in other directions, the apparent velocity will be as the cosine of the angle of inclination to the said perpendicular.

Plate 7.  
Fig. 14.

If two bodies set out at the same time from P, the one moving along the line PQ, perpendicular to the optic axis Qq, and the other along the long PS, oblique to it; and if their velocities be such, as to pass over the lines PQ, PS, in the same time, it is manifest that their apparent velocities will be the same; for the images of each will pass over the same space *pq*, on the *retina*, in the same time. The real velocities being, in this case, as PQ to PS, it is manifest, that, when the velocities are equal, the apparent velocity of the body which moves in PQ is to that of the body moving in PS, as PS to PQ, that is, as radius to the cosine of the angle QPS: but PS is always greater than PQ: whence the proposition is manifest.

## P R O P. LXXVIII.

If objects at different distances from the eye, move in parallel lines, nearly at right angles to the optic axis, and if their velocities are proportional to these distances, their apparent velocities will be equal;  
and



and if their real velocities are equal, their apparent velocities will be reciprocally as their distances from the eye.

Let an object move from Q to P, in the same time that another moves from G to H, their real velocities are as QP to GH, that is (El. VI. 2.) as QO to GO, the distances from the eye; and their apparent velocities will be equal, for the space  $qp$  upon the *retina* will be passed over in the same time by the image of each. If the velocities of the objects G, Q, be equal, the object G will arrive at K, and its image describe the space  $qk$  upon the *retina*, in the same time that the image of the object Q describes the space  $qp$ ; whence the apparent velocities of these two objects are as  $qk$  and  $qp$ , or as GK (or QP) and GH; that is (El. VI. 2.) the apparent velocity of the object G is to that of Q, as QO to OG.

Plate 7.  
Fig. 14.

SCHOL. It is here supposed, that the spectator makes no allowance for the different distances.

### P R O P. LXXIX.

The apparent velocities of bodies moving in parallel lines at different distances from the eye, are directly as the real velocities, and reciprocally as the distances.

Let two bodies move from Q, G, in the parallel lines QP, GH; let the velocity of the object Q be called V, and that of G,  $v$ ; and let their apparent velocities be called A,  $a$ . If the two objects be conceived to move in the same line GK, whatever be their velocities,  $V : v :: A : a$ ; and supposing the velocity of the object Q, to be the same in QP as before in GK, A in QP : A in GK :: GO : QO, by Prop. LXXVIII. and A in GK :  $a :: V : v$ ; whence, compounding these ratios,  $A : a :: GO \times V : QO \times v$ , that is,

Plate 7.  
Fig. 14.

$$:: \frac{V}{QO} : \frac{v}{GO}.$$

SCHOL. I. We judge of the *distance* of any object by the visible length of the plane which lies between the eye and the object. When this method fails us, we compare the known magnitude of the object, with its present apparent magnitude; or we compare the degrees of distinctness with which we see the several parts of an object; or we observe whether the change of the apparent place of an object when viewed from different stations, or its *parallax*, be great or small, this change being always in proportion to the distance of the object. On this principle, we may judge of the distance of a near object by observing the change which is made in its apparent situation, upon viewing it successively, with each eye singly. Or, since it is the difference of the apparent place of an object, as viewed by each eye separately, which makes an object to be seen double unless we turn both eyes directly



directly towards it, and since in doing this, where the distance is very small, we turn the eyes very much towards each other, and less at a greater distance; the different sensations accompanying the different degrees in which the eyes are turned towards each other, afford, by habit, a rule for judging of the distance of objects.

SCHOL. 2. In objects placed at such distances as we are used to, and can readily allow for, we know by experience how much an increase of distance will diminish their apparent *magnitude*, and therefore instantly conceive their real magnitude, and, neglecting the apparent, suppose them of the size they would appear if they were less remote; but this can only be done, where we are well acquainted with the real magnitude of the object; in all other cases, we judge of magnitudes by the angle which the object subtends at the known, or supposed, distance, that is, we infer the real magnitude from the apparent magnitude in comparison with the distance of the object.

## S E C T. II.

### *Of VISION as affected by REFRACTION.*

#### P R O P. LXXX.

When any small object is seen through a refracting medium, it appears in the direction of that line which the rays describe after their last refraction.

Plate 7.  
Fig. 16.

The ray DE from any small object D, in passing through a glass prism the end of which is ACB, will be refracted towards a perpendicular, and will describe the line EF; and when it goes out of the prism, it will be refracted from a perpendicular, into the line FG, which is its direction after its last refraction; and the object D will be seen in this direction at L instead of D. For the image in the *retina* will be in the place in which it would have been, if the naked eye had been looking at an object really placed at L, in the last direction of the rays.

Fig. 17.

COR. Hence an object seen through a glass cut into different surfaces inclined to one another, will appear at one view in many different places. The object A seen from the point F through the glass CEDB, will appear at H, A, G; the last direction of the rays AC, AD, AB, after refraction.

EXP. View any object through a multiplying glass.

SCHOL.



SCHOL. If a hair be placed across a small hole made in a thin board, and an eye situated in the dark look through the hole at a number of candles, the hair will appear multiplied: for a shadow of the hair is cast upon the eye by each of the candles.

## P R O P. LXXXI.

An object seen through a medium terminated by plain and parallel surfaces, appears nearer, brighter, and larger, than with the naked eye.

Let AB represent the object, CDEF the medium, and GH the pupil of an eye. Let RK, RL, be two rays proceeding from the point R, and entering the denser medium at K and L; these rays will here by refraction be made to diverge less (by Prop. XVII.) and to proceed afterwards, suppose in the lines Ka, Lb; at a and b, where they pass out of the denser medium, they will be as much refracted the contrary way, proceeding in the lines ac, bd, parallel to their first directions (by Prop. XI.) produce these lines back till they meet in e, this will be the apparent place of the point R, and it is evident from the figure, that it must be nearer the eye than that point; and because the same is true of all other pencils flowing from the object AB, the whole will be seen in the situation fg, nearer to the eye than the line AB. As the rays RK, RL, would not have entered the eye, but have passed by it in the directions Kr, Lt, had they not been refracted in passing through the medium, the object appears brighter. The rays Ah, Bi, will be refracted at b and i into the less converging lines bk, il, and at the other surface into kM, lM, parallel to Ah and Bi produced (by Prop. XI.) so that the extremities of the object will appear in the lines Mk, Ml, produced, namely, in f and g, and under as large an angle fMg, as the angle AqB, under which an eye at q would have seen it, had there been no medium interposed to refract the rays; and therefore it appears larger to the eye at GH, being seen through the interposed medium, than otherwise it would have done. But it is here to be observed, that the nearer the point e appears to the eye on account of the refraction of the rays RK, RL, the shorter is the image fg, because it is terminated by the lines Mf and Mg, upon which account the object is made to appear less; and therefore the apparent magnitude of an object is not much augmented by being seen through a medium of this form.

Farther, it is apparent from the figure, that the effect of a medium of this form depends wholly upon its thickness: for the distance between the lines Rr and ec, and consequently the distance between the points e and R depends upon the length of the line Ka: again, the distance between the lines AM and fM, depends on the length of the line bk; but both Ka and bk depend on the distance between the surfaces CE and DF, and therefore the effect of this medium depends upon its thickness.

P R O P.

Plate 7.  
Fig. 18.



## P R O P. LXXXII.

In viewing objects through a convex or concave lens, the object itself is not seen, but its last image, consisting of all the imaginary radiants from which the rays appear to diverge after the last refraction.

Plate 7.  
Fig. 8.

If AC be an object nearer the convex lens GIL than its principal focus, the rays which diverge from any point B in this object, will, in passing through the lens (by Prop. XVII.) be made to diverge less, and the imaginary radiant (by Prop. VI.) will be more distant than the real one. Hence the rays BG, BL, after refraction, will appear to diverge from the radiant E, farther from the lens than the real radiant B. The same happening to the rays from every other point of the object, there will be in DEF as many imaginary radiants as there are real radiants in the object ABC, which, taken together, will compose the last image. And since all the rays fall upon the eye as if they had diverged from this last image, the eye will be affected by the object ABC in the same manner in which it would be affected without the lens, by an object in all respects like DEF, that is, the eye perceives the last image.

## P R O P. LXXXIII.

An object seen through a convex or concave lens will appear erect, if the object and its image are on the same side of the lens, but inverted, if they are on contrary sides.

Plate 7.  
Fig. 8, 9.

It appears from the last Prop. that all the rays which diverge from B before refraction, will appear to diverge from E after refraction: and the like may be said of any other points, A and D, or C and F. Now AI is the axis of the beam which comes from A, and therefore, with the rest of the rays of the beam, after refraction, will appear to diverge from the point D in the same right line with A. The same may be shewn of FCI. Now these right lines only cross one another at the lens. Consequently, the highest point both of the object and image is in DI the upper side of the angle DIF, and the lowest points of both in FI, the lower side of the same angle; that is, the image, which is visible, is erect, or in the same position with the object.

If the object ABC is more remote than the principal focus of the convex lens E, there will be (by Prop. XXIV.) a distinct image formed on the other side of the lens. If the rays thus collected are not received upon a plane surface, they will proceed straight forwards; those which had converged diverging from the focus; whence an inverted image will be presented to an eye placed beyond the focus.—In the same manner this, and the preceding, proposition may be proved concerning an object seen through a concave lens.

Exp.



EXP. 1. Observe the image of a candle whose rays have passed through a convex lens, and are received at the focus on a white surface, whilst the object is on the same, or on the contrary side of the lens.

2. An inverted image will be produced without a lens, by solar rays passing through a very small hole into a darkened room; and if the edge of a knife be applied to one side of the hole, and moved slowly over it, the parts of the image situated opposite to the covered side will be first concealed; from whence it is manifest, that the rays cross one another in passing through the hole.

## P R O P. LXXXIV.

The apparent magnitude of an object seen through a lens placed close to the eye, or to the object, is equal to its apparent magnitude when seen without the lens.

If the eye be placed close to the lens at I, the diameter of the object of refracted vision DF, is to the diameter of the object of plane vision AC (by Prop. XXVIII.) as EI to IB, that is, as their respective distances from the vertex of the lens. Therefore (as appears from Prop. LXIX.) their apparent diameters when seen from I, are equal.

Plate 7.  
Fig. 8.

If the lens is placed close to the object, the real radiants touching the lens, the imaginary radiants, that is, the last image, will also touch the lens: whence their diameters, or apparent magnitudes, will be equal.

## P R O P. LXXXV.

If an object seen through a convex lens is nearer to the lens than its principal focus, it will appear brighter than to the naked eye, distinct, and in an erect position.

In this case, the rays which come from any point (by Prop. XVII.) will be brought nearer by refraction, and consequently a greater number will enter the eye than if there had been no refraction; whence it is manifest, that the object will appear brighter. Because the rays of each pencil diverge after refraction, but less than before, they come to the eye, as they would if the object were at a moderate distance and no lens were used, and therefore will be seen distinctly. And because the refracted rays (by Prop. XVII.) diverge less than the incident rays, that is (by Prop. VI.) the imaginary radiants are more remote than the real ones, the last image, as DEF, which is formed by these imaginary radiants, is farther from the lens than the object, and on the same side of the lens; and consequently, since the extreme axis DAI, FCI, only cross one another at the lens, the image will be in the same position with the object, and appear erect.

B b

P R O P.



## P R O P. LXXXVI.

If an object seen through a convex lens is farther from the lens than its principal focus, the object will appear brighter than to the naked eye, confused, and in an erect position.

If the eye be placed between the lens and the distinct image, whilst the eye is nearer the lens than the place of the image, the rays being made convergent by the lens, will be closer together, and therefore a greater number of them will enter the eye, and the object will appear brighter, than if it had been seen without a lens : because the rays come to the eye in a converging state, which from one and the same point they do not in plane vision, they will give a confused image. And because no image is formed till the rays come to the *retina*, the object will appear erect.

## P R O P. LXXXVII.

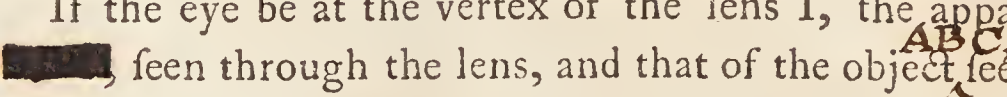
If an object seen through a convex lens is in the principal focus, it will appear brighter than to the naked eye, and erect.

The rays of each particular beam becoming in this case (by Prop. XVII.) parallel after refraction, are brought nearer together than if there had been no lens ; consequently, more rays will enter the eye from every point, and the object will appear brighter : and no image being formed before the rays come to the *retina*, the object will appear erect.

## P R O P. LXXXVIII.

When an object, seen through a convex lens, is nearer than the principal focus, it is magnified, unless the eye or the lens touches the object ; and as the eye departs from the lens, its apparent magnitude will decrease.

Plate 7.  
Fig. 8.

If the object ABC continues in the same place, or does not change its distance BI from the lens, the last image DF will (by Prop. XXVII.) remain at the same distance EI : therefore the real diameter of DF (by Prop. XXXIII.) will be invariable, wherever the eye is placed. If the eye be at the vertex of the lens I, the apparent diameters of the last image DEF, , seen through the lens, and that of the object <sup>ABC</sup> seen with the naked eye, are manifestly the same. Both those apparent diameters decrease as the eye recedes from the lens towards O ; but the apparent diameter of the last image decreases in the inverse ratio of OE to IE, and that of the object in the inverse ratio of OB to IB. But OE has a  
less



less ratio to IE than OB to IB; for IE and IB are unequal quantities, of which IE the distance of the image, is always (by Prop. XVII. and VI.) the greater, which are equally increased, but not proportionally: therefore the apparent diameter of the image decreases slower than that of the object, as the eye recedes from the lens. Consequently, when the eye is at any distance from the lens, the last image, or the object of refracted vision, will appear greater than the object seen by the naked eye. As the eye departs from the lens, the apparent magnitude of the object, from what has been said, must continually decrease, till at an infinite distance it vanishes.

## P R O P. LXXXIX.

If an object seen through a convex lens is more remote than the principal focus, and the eye on the other side of the lens is nearer than the place of the image, the object appears magnified, and its apparent magnitude will be inversely as the distance of the eye from the image.

Suppose the eye at the side of the lens GL; it might successively see both the object and the image without looking through the lens; and in this situation (by Prop. XXVIII.) the real diameter of the object is to that of the image, as their respective distances from the lens or the eye; consequently (by Prop. XXXIV.) their apparent diameters will be equal. Next, suppose the eye close to the image at F, E, or D, the apparent diameter of the image would manifestly be infinite. Also in this situation of the eye, the apparent diameter of the object would be infinite; for, if the eye be at F, the rays from the point C are the only rays collected into the eye, which appear diffused over the whole surface, and would do so if the lens were ever so large: and the same would be true of the points B or A, if the eye was at E or D; that is, the apparent diameter of the object seen through the lens is infinite. Since then the object and the image appear equal when the eye is close to the lens, and that both appear infinite when the eye is close to the image; they must have increased equally as the eye was moving from the lens to the image, and their apparent diameters must always have been equal. Hence, the object in every station of the eye when it does not touch the lens is magnified. And because the apparent diameter of the object seen through the lens is every where equal to that of the image, and that of the image (by Prop. XXXI.) inversely as the distance of the eye from the picture, the apparent diameter of the object seen through the lens, is inversely as the distance of the eye from the image.

Plate 7.  
Fig. 8.



## P R O P. XC.

If an object seen through a convex lens is placed in the principal focus, its apparent magnitude will not be altered by withdrawing the eye from the lens.

Since in this case the rays from the object come parallel to the eye, both the imaginary radiants and the image (by Prop. XVII.) are infinitely distant. Therefore the apparent magnitude of the object cannot be diminished by receding from the imaginary radiants, nor increased by approaching to the image, but will always remain the same.

## P R O P. XCI.

If a convex lens is moved whilst the eye and object remain fixed, the apparent magnitude of the object will increase, till the lens is at the middle point between them, after which it will decrease till the lens reaches the object; provided the eye is never farther from the lens than the place of the image.

When the lens is at either extreme (by Prop. LXXXIV.) the object is not magnified; but between the extremes (by Prop. LXXXVIII. and LXXXIX.) it is magnified; therefore when it is equally distant from the two extremes, it is most magnified, and must increase in its apparent magnitude as the lens moves from the eye towards the middle station, and decrease, as it moves from that middle station towards the object.

## P R O P. XCII.

The apparent magnitude of an object, seen through a concave lens, decreases as the eye, or the object, departs from the lens.

Plate 7.  
Fig. 9.

If the eye touches the vertex of the lens I, the apparent diameters of the object and the last image are equal. As the eye recedes from the lens, its distance both from the object ABC and last image DEF increases, and consequently, the apparent magnitude of both decreases. But the distance IE from the last image increases faster than the distance IB from the object, as was shewn in Prop. LXXXVIII. Therefore (by Prop. LXIX.) the apparent diameter of the last image, or the object of refracted vision, is diminished as the eye recedes from the lens, more than that of the object seen by the naked eye.— Again, as the object departs from the lens, the image departs with it; whence its visible diameter decreases.

P R O P.



## P R O P. XCIII.

When the eye and object are fixed, if a concave lens be moved from the eye, the apparent magnitude of the object will decrease till it reaches the middle point between them, and increase as it moves on towards the object.

When the lens is at each extreme, the apparent magnitude of the object seen through the lens (by Prop. LXXXIV.) is the same as when seen with the naked eye. In all other stations of the lens, the object appears diminished: therefore it must appear most of all diminished when the lens is in the middle station, and it must decrease whilst it is approaching to that station, and increase whilst it is departing from thence towards the object.

Exp. View a candle through a convex or concave lens, in the manner described Prop. XLII. varying the position of the object, or lens, according to the preceding Propositions, from Prop. LXXXIV.

## P R O P. XCIV.

Convex lenses assist the sight of those persons whose eyes are not sufficiently convex, and concave lenses, that of those whose eyes are too convex.

For convex lenses enable the former to bring the rays from objects to a focus nearer to the chryselline than can be done by their eyes; and concave lenses enable the latter to bring the rays to a focus at a greater distance; and thus to produce a distinct image upon the *retina*.

## S E C T. III.

*Of VISION as affected by REFLECTION.*

## P R O P. XCV.

If a plane mirror and the object seen in it are both perpendicular to the horizon, the object appears erect.

The object DE, and the mirror AB, being both perpendicular to the horizon, the lines DI, EZ, in which (by Prop. LVII.) the highest and the lowest points of the object

Plate 7,  
Fig. 7.



object appear, being both perpendicular to the surface AB, are parallel to each other, and do not meet. Therefore the line DI, which is highest at the object, is also highest at the image, and EZ will be lowest at both; therefore the image is not inverted with respect to the object; and each point of the image LM (by Prop. LIV.) is equally distant from the surface of the mirror with its corresponding point in the object DE: therefore LM, DE, are parallel, (El. I. 30.) and since the object is erect, the image will be so too.

## P R O P. XCVI.

When the object is parallel to a plane mirror, the length or breadth of that part of the mirror upon which the image appears, is to the length or breadth of the object, as any reflected ray is to the passage of reflection.

Plate 7.  
Fig. 7.

If the object DE is parallel to the mirror AB, and the image LM is seen by the eye at C, then FN, the length of that part of the mirror which is taken up by the image, subtends the angle LCM, under which the image appears. For since all the visible length of the image is manifestly included within the angle LCM, there cannot be more of the mirror taken up by that visible length, that is included within the same angle. Now the length of the image LM is equal (by Prop. LX.) to the length of the object DE. And (El. VI. 2.) FN is to LM, as FC to CL, or (by Prop. LIX.) CFD; that is, the length of that part of the mirror which is taken up by the image is to the length of the image, or (by Prop. LX.) the length of the object, as any reflected ray is to the passage of reflection of that ray. In the same manner it may be shewn, that the breadth of that part of the mirror taken up by the image, is to the breadth of the object in the same ratio.

COR. Hence, in plane mirrors the object, and the part of the surface on which it appears, are similar.

## P R O P. XCVII.

If, at a certain distance from the mirror, the whole object cannot be seen by reflection, the whole will become visible either by bringing the eye nearer to the mirror, or removing the object farther from it.

For (by Prop. XCVI.) the less the ratio of the reflected ray is to the passage of reflection, so much the less will be the ratio of the length of that part of the mirror on which the whole object will appear, to the length of the object. If therefore the reflected ray



ray CF decreases by bringing the eye nearer to the mirror, since it is diminished faster than the greater quantity DFC, the passage of reflection, when equal parts are taken from each, the ratio of the reflected ray CF to the passage of reflection DFC (El. V. 8.) diminishing, the ratio of the length of that part of the mirror on which the whole object will be seen to the object, is diminished; that is (the object being given) the length of the part of the mirror on which the whole object is seen, will be diminished: consequently, the whole object, which at a certain distance of the eye from the mirror was not visible, on a nearer approach of the eye may become visible.

Plate 7.  
Fig. 7.

If the object DE be removed farther from the glass, the ratio of the reflected ray FC to the passage of reflection DFC will also be diminished, because DFC will be increased whilst FC remains the same; and consequently the length of the part of the mirror on which the whole image is seen, is diminished, and a less surface of glass is required in order to see the whole image.

#### P R O P. XCVIII.

If a spectator sees himself entirely in a plane mirror placed parallel to him, the mirror must be half as long as himself.

When a spectator is looking at himself, the incident ray is his distance from the mirror, and the reflected ray is equal to it, and is the distance of the mirror from him. The passage of reflection is therefore equal to twice his distance from the mirror; and consequently the reflected ray is to the passage of reflection as 1 to 2: whence (by Prop. XCVI.) the length of the glass, in which he can see himself entirely, must be to his own length as 1 to 2, or the length of the glass must be half his own length.

If the mirror be at all shorter than this, the spectator will not be able to see himself, whether he is nearer to the glass, or farther from it. If he approaches towards the glass, the object, being himself, approaches as fast as the eye, so that though (by Prop. XCVII.) he might see more of himself by the approach of the eye, he will see just as much less of himself on account of the approach of the object. In the same manner it may be shewn, that if he recedes from the mirror, he will not be able to see himself entirely.

#### P R O P. XCIX.

Objects perpendicular to the horizon, seen in a plane mirror parallel to the horizon, appear inverted.

By Prop. LIV. each imaginary radiant is at the same distance behind the mirror, that the real radiant is before it: hence, if the mirror be below the object and the eye, the object will have its lowest part nearest the surface of the mirror, and its highest part  
farthest



farthest from it, and therefore will in this situation appear inverted: if the mirror be above the object and the eye, the object will have its highest part nearest the surface, and its lowest part farthest from it, and therefore will, in this situation also, appear inverted.

## P R O P. C.

If a plane mirror is inclined to the horizon at an angle of forty-five degrees, an object parallel to the horizon will appear erect in the mirror, and an object perpendicular to the horizon will appear parallel to it.

Plate 7.  
Fig. 19.

Let the object  $CD$  parallel to the horizon be seen in  $AB$ , a mirror so placed as to incline to the horizon in an angle of forty-five degrees. At whatever distance any radiant  $C$  is from the mirror, at the same distance (by Prop. LIV.) is the corresponding radiant  $C$  in the image, or  $CE$  will be equal to  $cE$ . In like manner,  $DB$  will be equal to  $dB$ . Thus every radiant in the image is at the same distance behind the mirror, as the object is before it; whence the image  $cd$  makes half a right angle  $cBA$  with the mirror on one side, whilst the object makes with it half a right angle  $CBA$  on the other: whence  $cBC$  is a right angle, that is, the image is perpendicular to the object or horizon, and appears erect in the mirror.

By making  $cd$  the object, and  $CD$  the image, it may be shewn in like manner that when the object is erect, it will appear ~~parallel~~ in the mirror parallel to the horizon.

## P R O P. CI.

If an object is placed between two plane mirrors inclined to one another at any angle, several images may be seen.

Plate 7.  
Fig. 20.

Let the object  $F$  be placed between the two plane mirrors  $CB$ ,  $CA$ , making with one another the angle  $BCA$ . From the object  $F$ , draw  $FD$  perpendicular to the mirror  $CA$ , meeting  $CA$  in  $K$ , and make  $KD$  equal to  $FK$ . The image of  $F$  will (by Prop. LIV.) appear at  $D$ . In like manner, if  $FG$  be drawn perpendicular to  $CB$ , the object will be seen in  $G$ , as far behind the mirror as  $F$  is before it. Thus two images of the same object, but of different sides or surfaces of it, will be seen.

Again, since some of the reflected rays, which diverge from the image  $D$  in all directions fall upon the opposite mirror  $CB$ , the image  $D$  may be considered as an object placed before the mirror  $CB$ : and consequently, when the rays which diverge from  $D$  are reflected from  $CB$ , if  $DHE$  be drawn perpendicular to  $CB$ , and if  $EH$  is taken equal to  $HD$ , these reflected rays will (by Prop. LVIII.) represent the image of  $D$  at  $E$ ,



as far behind the mirror as *D* is before it. In like manner, some of the rays from this second image *E* will fall upon the mirror *CA*, and the image of *E*, or a third image of the object, will appear. And thus, as long as the image represented in one mirror is before the other, so long a new image of the last image will be produced. And all these images, beginning from *CA*, and being successive representations of *D*, will be images of the side of the object *F* towards *CA*. Besides these, there will be another set of images beginning from *CB*, which will be formed in the same manner and represent the side of the object towards *CB*, the first of which will be *G*, and the second *L*.

## P R O P. CII.

The images which appear in two plane mirrors inclined to one another, are in the circumference of a circle, the radius of which is the distance of the object from the vertex of the angle contained between the mirrors.

Since *FK* is equal to *KD*, *KC* common, and the angles at *K* right angles, *CF*, the distance of the object from the vertex of the angle made by the inclination of the mirrors to one another, is (El. I. 4.) equal to *CD*. In like manner it may be proved that *CE* is equal to *CD*. Therefore *CF* and *CE* are equal to one another. Thus all the straight lines drawn from *C* to *G*, *L*, or any other image in the mirrors, may be proved to be equal to *CF*. Consequently, if *C* be made the center of a circle, and *CF* its radius, the circumference will pass through the points *D*, *E*, *G*, *L*, and every other image which appears in the mirrors, that is, all the images, are in the circumference of a circle, whose radius is the distance of the object from the vertex of the angle made by the inclination of the mirrors to one another.

Plate 7.  
Fig. 20.

## P R O P. CIII.

If two plane mirrors are parallel to one another, and an object is placed between them, innumerable images of that object may be seen in each, standing in a right line.

If the two mirrors *CA*, *CB*, were separated at *C*, so as to be less inclined, or nearer parallel to one another, the angle of inclination being diminished, it is manifest from Prop. CII. that the number of images will be increased. At the same time the circumference of the circle in which the images are placed will be enlarged, because the vertex *C* is farther removed from *F*, or *FC* is increased. Consequently, if the mirrors *CA*, *CB*, are so far separated, that the vertex is infinitely distant, the images become innumerable, and they are placed in a straight line.

Plate 7.  
Fig. 20.

C c

P R O P.



## P R O P. CIV.

In spherical mirrors, concave or convex, when the place of the image is determinate, the object and the image are in the same situation, if they are both on the same side of the center, and in contrary situations if they are on opposite sides.

Plate 7.  
Fig. 23.

Let AFB be an object placed nearer the concave mirror SGV than its principal focus, and let C be the center of concavity. The rays from A, F, B, being reflected, will (by Prop. LII.) diverge, and the distances of the corresponding imaginary radiants I, E, M, may be determined by Prop. LVI. The real and the imaginary radiants are, in this case, on the same side of C the center of concavity. Now, the imaginary radiant which corresponds to the real one A is (by Prop. LVIII.) in CAI the perpendicular drawn from A to the surface; and the same with respect to B, and all the other radiants. And these perpendiculars, and all the rest, being drawn from the center, do not cross each other but at the center; consequently they are in the same position with respect to each other at the object and the image, and that which is the highest at the object will be the highest at the image, and the reverse. Since therefore (by Prop. LVIII.) every point of the object appears in its perpendicular at the image, the highest point in the object will appear the highest in the image, and the reverse; that is, the object and image will be in the same situation. In like manner it may be shewn, that if the object AFB is placed before a convex mirror, and its image IM is on the same side of the center, they will be both in the same situation. If the object AFB is farther from the concave mirror SGV than its principal focus, and if M, E, I, be the places of the several foci, to which the rays from A, F, B, (by Prop. L.) will converge, a distinct image of the object will appear upon a paper placed at M, E, I: and if the paper is taken away, and the eye is more remote from the mirror than MEI, the rays will diverge from these foci, and become the last image. But, because the extreme perpendiculars ICH, MCK, in which (by Prop. LVIII.) the points A and B will appear, cross each other at the center C, between the object and image, A the highest point of the object will appear at I the lowest point of the image, and the reverse; that is, the image with respect to the object will be inverted, or they will be in contrary situations. The same may be shewn in like manner with respect to the convex mirror.

Plate 7.  
Fig. 22.

Fig. 21.

## P R O P. CV.

In spherical mirrors, concave or convex, the diameter of the object is to the diameter of the image, as the distance of the object from the center to the distance of the image from the center, and also as the



the distance of the object from the surface to the distance of the image from the surface.

If the eye is any where in the line  $FG$ , or that line produced,  $FG$  is the optic axis; whence the visible length of the object  $AB$ , and also of the image  $IM$ , is proportional (as appears from Prop. LXVIII.) to a subtense of the optic angle perpendicular to  $FG$ . The visible extensions, or lengths of the object and of the image being then perpendicular to the same line  $FG$ , are parallel to one another. Hence in all the cases, the angles  $ACB$ ,  $ICM$ , are equal, and also the angles  $CAB$ ,  $CIM$ . Therefore (El. VI. 4.)  $AB$  the visible length of the object is to  $MI$  that of the image, as  $AC$  the distance of the object from the center is to  $IC$  the distance of the image.

Plate 7.  
Fig. 21, 23.

Again, since the object  $AFB$  consists of real radiants, and the image  $MEI$  of imaginary radiants when the rays diverge, and of foci when they converge, after reflection; and since when they diverge,  $FG$  the distance of the object from the surface is (by Prop. LVI.) to  $EG$  the distance of the image from the surface, as  $FC$ , the distance of the object from the center is to  $EC$  the distance of the image from the center; but (by El. VI. 2.)  $AB$  is to  $MI$ , as  $FC$  to  $EC$ : therefore  $AB$  is likewise to  $MI$  as  $FG$  to  $EG$ , that is, the diameter of the object is to that of the image, as the distance of the object from the surface to that of the image from the surface.

#### P R O P. CVI.

If the eye is close to a concave or convex mirror, the apparent diameter of the object is equal to the apparent diameter of the image.

If the eye is at  $G$ , the real diameters of the object  $AFB$  and image  $MEI$  are (by Prop. CV.) as their respective distances from the eye. Therefore (by Prop. LXXI.) their apparent diameters will subtend the equal angles  $AIC$ ,  $DIF$ , and will be equal.

#### P R O P. CVII.

If the eye is placed in the center of a concave mirror, it can see nothing in the mirror but its own image.

For the eye, in this situation, is in the place of its own image, and therefore rays will be reflected to it from every point of the surface.



## P R O P. CVIII.

If an object is nearer to a concave mirror than its principal focus, the image appears behind the mirror, farther from the mirror and larger than the object, erect, and distinct.

Plate 7.  
Fig. 23.

The object AFB being nearer to the concave mirror than the principal focus, the rays which diverge from each point in the object before the mirror will diverge after reflection (by Prop. L.) less than before from the imaginary radiants M, E, I; whence the image formed by them will (Prop. VI.) be farther from the mirror than the object; and consequently (by Prop. CV.) it will be larger than the object. Because the object is nearer the mirror than its principal focus, it is likewise nearer than the center; whence (by Prop. CIV.) the image will be erect. Lastly, the image will be seen distinctly, because the rays from it diverge as from objects at a moderate distance.

## P R O P. CIX.

If an object touches a concave mirror, the image will touch it likewise, and they will be equal.

For the real radiants being close to the mirror, the imaginary radiants are so too; whence (by Prop. CV.) their diameters will be equal.

## P R O P. CX.

If an object is placed in the focus of a concave mirror, the image is at an infinite distance behind the mirror, larger than the object, erect, and distinct.

When the object AFB is in the principal focus of the concave mirror SGV, the image is at an infinite distance. It will (by Prop. CV.) be larger than the object, on account of its remoteness. It will be seen erect (by Prop. CIV.) because it is on the same side of the center with the object. And since the rays of each beam are parallel, it will be seen as distinctly as the naked eye sees very remote objects.

P R O P.



## P R O P. CXI.

If the object is farther from a concave mirror than its focus, and the eye is nearer than the place of the image, the object will appear confused, behind the mirror, erect, and magnified.

The rays which diverge from A, F, B, in an object more remote from the concave mirror SGV than its focus, will (by Prop. L.) be collected, and on a surface of white paper form an inverted image. If the eye is any where between B and G, the rays from every radiant are converging when they come from the mirror to the eye, whence it will appear confused, because the eye is not accustomed to see rays in this state. The rays AH, AG, diverging from A, will, after reflection, converge towards I: but if the eye is nearer to the mirror than I, the reflected ray GI will not cross its perpendicular HI in any place before the eye, since they are in a state of convergency; consequently, the apparent place of this, or any other point of the image, will be indeterminate. It will be seen erect, because MEI, the inverted image of the object, will be drawn inverted on the *retina*. Lastly, because the rays of each beam converge after reflection, as HI, GI, they will appear to come, not from points, but from circular spots larger than the points of the object, the image will appear confused, and be larger than the object.

Plate 7.  
Fig. 21.

## P R O P. CXII.

If an object is farther from a concave mirror than its principal focus, and the eye is farther from the mirror than the place of the image, the image appears before the mirror, inverted, and distinct.

Rays coming from M, E, I, an object at a greater distance than the principal focus from the mirror SGV, will (by Prop. L.) converge to AFB, and would paint an inverted image upon a surface of white paper placed there. From thence they will diverge, (the paper being taken away); whence, to the eye placed any where beyond AFB, the image will appear inverted. And it will be seen distinctly, because the rays come to the eye diverging, as from an object at a moderate distance.

Plate 7.  
Fig. 22.

SCHOL. 1. The inverted images of objects may be represented in a dark room by a concave mirror, which receives rays passing from external objects through a hole in a window-shutter, and collects them into a focus on a surface of white paper.

SCHOL. 2. A concave mirror, collecting the parallel rays of the sun into a focus, will act as a burning glass.

P R O P.



## P R O P. CXIII.

When an object is placed before a convex mirror, its image appears behind the mirror, nearer the mirror, and less than the object, distinct, and erect.

Plate 7.  
Fig. 22.

If AFB is an object placed before the convex mirror SGV, the rays which before reflection diverge from A, F, B, will (by Prop. LVIII.) after reflection diverge from as many radiants I, E, M, behind the mirror, forming the image. And because the reflected rays (by Prop. LII.) diverge more than the incident ones, the image IM (by Prop. VI.) will be nearer the mirror than the real radiants, or object AFB: whence (by Prop. CV.) the image will be less than the object. And because the reflected rays come to the eye in a state of divergency, the image will be seen as distinctly as any visible object, seen by such diverging rays, at the same distance. Lastly, if the object AFB were at an infinite distance from the mirror, the rays proceeding from any point in it would fall parallel upon the mirror, and therefore would upon reflection form the image in the principal focus, that is, in the middle point between G and C, or on the same side of C with the object; whence (by Prop. CIV.) it would be erect. At any finite distance of the object, the image, being still nearer the surface, must therefore be erect.

## P R O P. CXIV.

When either the eye or the object departs from a convex mirror, the apparent diameter of the image decreases.

Plate 7.  
Fig. 22.

If the object AFB continues in its place, the image IM will (by Prop. LVI.) be always at the same distance from the mirror; and (by Prop. CV.) the real diameter will be invariable: consequently (by Prop. LXIX.) the apparent diameter of the image will be inversely as the distance of the eye.

If the object AFB departs from the mirror, the ratio of FG, the distance of the object, to GE, that of the image, (El. V. 8.) will increase: whence (by Prop. CV.) the ratio of the diameter of the object to that of the image will likewise increase, that is, the image will become less with respect to the object: but, the eye remaining in the same place, the apparent diameter of the image will (by Prop. LXX.) be as its real diameter: consequently, the apparent diameter will decrease.

Exp. The Propositions in this section may be confirmed, by placing an object before a Plane, Concave, or Convex Mirror, according to the terms of the respective Propositions.

C H A P.



## C H A P. V.

## O F C O L O U R S.

## S E C T. I.

*Of the different REFRACTIBILITY of LIGHT.*

DEF. XXV. Rays of light are *differently refrangible*, when at the same or equal angles of incidence, some are more turned out of the way than others.

DEF. XXVI. Rays are *differently reflexible*, when some are more easily reflected than others.

DEF. XXVII. Light is called *homogeneous*, when all the rays are equally refrangible; and *heterogeneous*, when some rays are more refrangible than others.

DEF. XXVIII. The *Colours* of homogeneous rays, are called *primary* or *simple* colours; those of heterogeneous, *secondary* or *mixed*.

## P R O P. CXV.

The rays of the sun are not all equally refrangible; and those rays which have a different degree of refrangibility, have likewise a different colour.

If a beam of light SF from the sun passes into a dark room through F a round hole in a window-shutter EG, and is received upon a white surface, a white round image will be seen. If a glass prism ABC is so placed as to receive the beam of light, the rays of this beam, from their refraction in passing through the prism, will be turned upwards, and the

Plate 7.  
Fig. 24.



the refracted image  $PT$  will be oblong, having its breadth equal to the diameter of the circular picture  $O$ . If all the rays were equally refracted upwards, it is manifest that such a refraction would not change the form of the picture. Since therefore the refracted image is oblong, it must be formed by rays differently refrangible, which fall with equal angles of obliquity upon  $BC$  the first side of the prism, but are some of them, in refraction, turned more out of the way than others; those rays which go to  $P$ , the upper part of the image, being most refrangible, and those which go to  $T$ , the lower part, being least refrangible.

This oblong image is of different colours in different parts; the whole image being made up of rays of seven different colours, in the following order, beginning with those which are most refrangible; violet, indigo, blue, green, yellow, orange, red. This refracted picture consists of several round pictures so near each other, that each higher circle mixes in part with that below it, whence the colours near the upper and lower edge of each circle are blended. The sides of these circles being very near to each other appear like right lines.

EXP. I. Observe the prismatic image formed by the refraction of the rays in passing through a single prism.

Plate 7.  
Fig. 27.

2. To separate the several colours as much as possible, make the hole  $F$  in the window-shutter very small, and collect the rays which pass through it, into a focus  $L$ , by a convex lens  $MN$ . Let the rays which have passed through the lens be now received upon a prism placed near the lens; the rays will be refracted upwards into an oblong image.

Plate 7.  
Fig. 25.

3. To prove that the prismatic image is produced by the different refrangibility of the rays, and by no other cause, let a second prism  $DH$  be placed beyond the first  $abc$ , at right angles to it. The rays passing through this second prism are refracted sideways; those which were most refracted upwards by the first prism, are most refracted sideways by the second; but, the rays not being spread in breadth, the image remains of the same form.

#### P R O P O S I T I O N CXVI.

Those rays of light which are most refrangible, are also most reflexible.

Plate 7.  
Fig. 26.

If the beam of light passing through  $F$  falls upon a prism  $ABC$ , whose sides  $AC$ ,  $AB$ , are equal, and the angle at  $A$  a right angle; when the obliquity of these rays, as they are to



to pass out of the prism at its base BC, is less than 40 degrees, the greatest part of the beam will pass through, but some rays will be reflected at the surface BC. Those rays which pass through the base (by Prop. CXV.) form an oblong coloured image at HG, between the most refrangible ray MH, and the least refrangible ray MG. If the rays which are reflected from M are made to pass through another prism XYV, they will also form a faint oblong coloured image *pt*. Now, if the prism ACB is turned slowly round upon its axis in the direction ACB, the obliquity of the rays FM to the base BC, will continually increase, till all the rays will be reflected at M. Consequently, the image *pt* will become much brighter than before. And this total reflection will not be produced at once, but the most refrangible rays MH will be first entirely reflected; for the violet colour in HG will first disappear, and the same colour at *p* will first become brighter. In like manner, as the prism ABC is turned round, each different sort of ray will be reflected sooner, as it has a greater degree of refrangibility. Hence it appears, that the rays of the sun have different degrees of reflexibility, and that those which are most refrangible are also most reflexible.

## P R O P. CXVII.

Homogeneous light is refracted regularly without any dilatation of the rays.

EXP. When the rays of any colour in the oblong image, as green, are separated from the rest, in the manner described in Prop. CXV. if some of these rays are transmitted through a small hole in a thin board, and refracted by a prism placed in the other side, the image formed by these rays after refraction will not be oblong, but circular.

## P R O P. CXVIII.

The confused appearance of objects seen through refracting bodies, is owing to the different refrangibility of light.

EXP. Small objects placed in a sun beam and viewed through a prism, will be seen confusedly; but if they are placed in a beam of homogeneous light separated by a prism, they will appear as distinct through a prism, as when viewed by the naked eye.

SCHOL. I. Although the 13th Proposition (in which it was shewn that when a ray of the sun is passing out of one medium into another, the ratio of the sine of incidence to the refracted sine will not be changed by changing the obliquity of the incident ray) proceeds upon the supposition that all rays are equally refrangible, and therefore is not  
D d exactly



exactly true; the demonstration is strictly applicable to any one sort of rays, as the red ones, which are equally refrangible.

SCHOL. 2. Since, all other circumstances being equal, the same cause, namely, the passing of the rays out of one given medium into another, will turn the violet rays more out of the way than the red rays; the attracting force which acts upon both being the same, it is probable that any single ray of the least refrangible sort contains a greater quantity of matter than any single ray of the most refrangible sort.

### P R O P. CXIX.

The colours of homogeneous light can neither be changed by refraction nor reflection.

EXP. 1. Let a beam of homogeneous light pass through a round hole in a pasteboard, and then be refracted by a prism on the other side, the colour of the rays will remain the same.

2. Red lead, viewed in homogeneous red light, will be red, but if placed in green, or any other homogeneous light, it will take the colour of the rays which fall upon it.

### P R O P. CXX.

The whiteness of the sun's light arises from a due mixture of all the primary colours.

Plate 7.  
Fig. 28.

EXP. If the oblong picture PT fall upon the convex lens MN, the rays, which were separated at PT, will, by passing through the lens, be collected into a focus at G, and form a round image of the sun upon a piece of paper DE. This image formed of all the primary sorts of rays, is white. That the whiteness of the image is owing to the due mixture of all the sorts of rays, appears from hence, that, if any of the colours be intercepted at the lens, the image loses its whiteness. The paper being removed from DE to *de*, the rays, having crossed at G, will form the prismatic image *tp*, inverted, but distinct; from whence it appears, that the colours are not changed by being mixed at the focus.

### P R O P. CXXI.

The colours of all bodies are either the simple colours of homogeneous light, or such compound colours as arise from a mixture of homogeneous light.

Each



Each sort of light having a peculiar colour of its own, which no refraction or reflection can alter, since bodies appear coloured only by reflected light, their colours can be no other than the colour of some single homogeneous light, or of a mixture of different sorts of light.

## P R O P. CXXII.

Water, air, glass, or any other transparent substance, when drawn into thin plates, become coloured.

EXP. I. If a soap-bubble be blown up, and set under a glass that the motion of the air may not affect it, as the water glides down the sides and the top grows thinner, several colours will successively appear at the top, and spread themselves from thence in rings down the side of the bubble, till they vanish in the same order in which they appeared. At last a black spot appears at the top, and spreads till the bubble bursts.

2. If a piece of plane polished glass is placed upon the object glass of a long telescope, and the interval between them is filled up with water, as the glasses are pressed together the same colours arise at the point of contact, and spread themselves in circular rings round that point in the same order as in the soap-bubble.

SCHOL. The *opacity* of bodies is owing to the numerous reflections and refractions which rays of light suffer within the bodies.

Almost all natural bodies are transparent, if made exceedingly thin, or reduced to very small particles. Hence it appears, that every single particle transmits light, or is transparent; and consequently, that the whole would transmit light, unless the rays, when they are to pass through all the particles which make up the whole, were so turned out of the way by innumerable refractions and reflections, as to be stopped in their passage.

The medium with which the pores of opaque bodies are filled, is not of the same density with the particles of those bodies. For the light would, in that case, be neither refracted nor reflected in passing out of the particles into the interstices, and the body would be transparent.

EXP. Paper wetted with water or oil is more transparent than when dry, because the pores are then filled with a fluid nearly of the same density with the paper.



## S E C T. II.

*Of the RAINBOW.*

## P R O P. CXXIII.

When the rays of the sun fall upon a drop of rain and enter into it, some of them, after one reflection and two refractions, may come to the eye of a spectator, whose back is towards the sun, and his face towards the drop.

Plate 8.  
Fig. 1.

If the sun shines upon XY, a drop of rain, in any lines SF, SD, SA, &c. the greatest part of the rays will enter the drop, and passing on to the second surface, will be transmitted through the drop. But at PG in the second surface some few rays will be reflected, and proceed in some such lines as NR, NQ; and coming out of the drop in the lines RV, QT, they may fall upon the eye of a spectator, placed in those lines with his face towards the drop. These rays are refracted when they enter the drop, reflected from the second surface, and again refracted when they come out of the drop.

DEF. XXIX. When rays of light reflected from a drop of rain come to the eye, those rays which excite a perception of light, are called *effectual*.

## P R O P. CXXIV.

When rays of light come out of a drop of rain, they will not be effectual, unless they are parallel and contiguous.

Most of the rays, which enter the drop between X and A, passing out of the hinder surface between P and G, only a few rays are reflected, and come out of the drop through the nearer surface between A and Y. Now of these, only the rays which are parallel to one another will be effectual, because if they diverge they will be so far asunder when they come to the eye, that only a very few of them can enter the pupil, and no perception of colours will be excited. Also, unless several parallel rays be very near each other, the rays will be too few to create any perception.

P R O P.



## P R O P. CXXV.

When rays of light come out of a drop of rain after one reflection, those will be effectual which are reflected from the same point, and entered the drop near one another.

Any rays AB, CD, when they have passed out of the air into a drop of water, will be refracted towards the perpendiculars BL, DL, by Prop. XI. And as the ray AB falls farther from the axis than the ray CD, AB will be more refracted than CD: so that these rays, though parallel to one another at their incidence, may describe the lines BE, DE, after refraction, and be both of them reflected from one and the same point E. Now all rays, which are thus reflected from one and the same point, when they have described the lines EF, EG, and after reflection emerge at F and G, will be so refracted, when they pass out of the drop into the air, as to describe the lines FH, GI, parallel to one another. If these rays were to return from E in the lines EB, ED, and were to emerge at B and D, they would be refracted into the lines of their incidence BA, DC, (by Prop. XII.) But if these rays, instead of being returned in the lines EB, ED, are reflected from the same point E in the lines EG, EF, the lines of reflection EG and EF will be inclined both to one another and to the surface of the drop, just as much as the lines EB and ED are. First, EB and EG make just the same angle with the surface of the drop; for the angle BEX, which EB makes with the surface of the drop, is the complement of incidence, and the angle GEY, which EG makes with the surface, is the complement of reflection; and these two are equal to one another, by Prop. XLV. In the same manner we might prove that ED and EF make equal angles with the surface of the drop. Secondly, the angle BED is equal to the angle FEG, or the reflected rays EG, EF, and the incident rays BE, DE, are equally inclined to each other. For the angle of incidence BEL is equal to the angle of reflection GEL, and the angle of incidence DEL is equal to the angle of reflection FEL, by Prop. XII. Consequently, the difference between the angles of incidence is equal to the difference between the angles of reflection, or  $BEL - DEL$  is equal to  $GEL - FEL$ , or BED to GEF. Since therefore either the lines EG, EF, or the lines EB, ED, are equally inclined both to one another and to the surface of the drop, the rays will be refracted in the same manner, whether they were to return in the lines EB, ED, or are reflected in the lines EG, EF. But if they were to return in the lines EB, ED, the refraction, when they emerge at B and D, would make them parallel. Therefore if they are reflected from one and the same point E in the lines EG, EF, the refraction, when they emerge at G and F, will likewise make them parallel.

Farther, in order to render the rays which emerge at F and G effectual, they must not only emerge in a direction parallel to each other, but must enter the drop nearly at the same place.

Let

Plate 8.  
Fig. 2.



Plate 8.  
Fig. 1.

Let  $XY$  be a drop of rain,  $AG$  the axis or diameter of the drop, and  $SA$  a ray of light, that comes from the sun and enters the drop at the point  $A$ . This ray  $SA$ , because it is perpendicular to both the surfaces, will pass straight through the drop in the line  $AGH$  without being refracted, by Prop. XIV. But any collateral rays that fall about  $SB$ , as they pass through the drop, will be made to converge to their axis, and passing out at  $N$  will meet the axis at  $H$ , by Prop. XI. Rays which fall farther from the axis than  $SB$ , such as those which fall about  $SC$ , will likewise be made to converge; but then their focus will be nearer to the drop than  $H$ , by Prop. XI. Suppose therefore  $I$  to be the focus to which the rays that fall about  $SC$  will converge; any ray  $SC$ , when it has described the line  $CO$  within the drop, and is tending to the focus  $I$ , will pass out of the drop at the point  $O$ . The rays, that fall upon the drop about  $SD$  more remote still from the axis, will converge to a focus still nearer than  $I$ , suppose at  $K$ , by Prop. XXI. note. These rays therefore go out of the drop at  $P$ . The rays, that fall still more remote from the axis, as  $SE$ , will converge to a focus nearer than  $K$ , as suppose at  $L$ ; and the ray  $SE$ , when it has described the line  $EO$  within the drop, and is tending to  $L$ , will pass out at the point  $O$ . The rays, that fall still more remote from the axis, will converge to a focus still nearer. Thus the ray  $SF$  will, after refraction, converge to a focus at  $M$ , which is nearer than  $L$ , and having described the line  $FN$  within the drop, it will pass out at the point  $N$ . Now here we may observe, that as any rays  $SB$  or  $SC$  fall farther above the axis  $SA$ , the points  $N$  or  $O$ , where they pass out behind the drop, will be farther above  $G$ , or, that as the incident ray rises from the axis  $SA$ , the arc  $GNO$  increases, till we come to some ray  $SD$ , which passes out of the drop at  $P$ , and this is the highest point where any ray, that falls upon the side  $AX$ , can pass out: for any rays  $SE$ , or  $SF$ , that fall higher than  $SD$ , will not pass out in any point above  $P$ , but at the points  $O$ , or  $N$ , which are below it. Consequently, though the arc  $GNOP$  increases, whilst the distance of the incident ray from the axis  $SA$  increases, till we come to the ray  $SD$ ; yet afterwards the higher the ray falls above the axis  $SA$ , this arc  $PONG$  will decrease.

As there are many rays which pass out of the drop between  $G$  and  $P$ , so, by Prop. XLIII. some few rays will be reflected from thence; and consequently the several points between  $G$  and  $P$ , which are the points where some of the rays pass out of the drop, are likewise the points of reflection for the rest, which do not pass out. Therefore in respect of those rays which are reflected, we may call  $GP$  the arc of reflection, and may say that this arc of reflection increases, as the distance of the incident ray from the axis  $SA$  increases, till we come to the ray  $SD$ ; the arc of reflection is  $GN$  for the ray  $SB$ , it is  $GO$  for the ray  $SC$ , and  $GP$  for the ray  $SD$ . But after this, as the distance of the incident ray from the axis  $SA$  increases, the arc of reflection decreases; for  $OG$ , less than  $PG$ , is the arc of reflection for the ray  $SE$ , and  $NG$  is the arc of reflection for the ray  $SF$ .

From hence it is obvious, that some one ray, which falls above  $SD$ , may be reflected from the same point with some other ray, which falls below  $SD$ . Thus, for instance, the ray  $SB$  will be reflected from the point  $N$ , and the ray  $SF$  will be reflected from the same point; and consequently, when the reflected rays  $NR$ ,  $NQ$ , are refracted as they pass out

of



of the drop at R and Q, they will be parallel, by what has been shewn in the former part of this Prop. But since the intermediate rays, which enter the drop between SF and SB, are not reflected from the same point N, these two rays alone will be parallel to one another when they come out of the drop, and the intermediate rays will not be parallel to them. And consequently these rays RV, QT, though they are parallel, after they emerge at R and Q, will not be contiguous, and for that reason will not be effectual, by Prop. CXXIV. The ray SD is reflected from P, which has been shewn to be the limit of the arc of reflection; such rays, as fall just above SD and just below SD, will be reflected from nearly the same point P, as appears from what has been already shewn. These rays therefore will be parallel, because they are reflected from the same point P; and they will likewise be contiguous, because all of them enter the drop at one and the same place, very near to D. Consequently such rays, as enter the drop at D and are reflected from P the limit of the arc of reflection, will be effectual, by Prop. CXXIV. since when they emerge at the part of the drop between A and Y, they will be both parallel and contiguous.

## P R O P. CXXVI.

When rays which are effectual emerge from a drop of rain after one reflection and two refractions, those which are most refrangible will, at their emerfion, make a less angle with the incident rays than those do which are least refrangible; by which means, the rays of different colours will be separated from one another.

Let FH, GI, be effectual violet rays emerging from the drop at F, H; and FN, GP, effectual red rays emerging from the same drop at the same points. The violet rays (by Prop. CXXIV.) are parallel among themselves, because they are effectual: for the same reason the red rays are parallel among themselves: but on account of the difference of refrangibility of the violet and red rays, the violet ray GI is not parallel to the red ray GP, but they diverge from the point G; and so of the rest. Both the violet ray GI and the red ray GP are refracted from the perpendicular LO, but (by Prop. CXV.) GI more than GP; whence the angle IGO is greater than the angle PGO. If the incident ray AB be continued in the direction ABK, and if IG and PG be continued backward till they meet AB in K and W, the angle IKA is that which the violet or most refrangible ray makes at its emerfion with the incident ray, and PWA that which the red or least refrangible ray makes with the same. And the angle IKA (El. I. 16.) is less than the exterior angle PWA. The same may be proved concerning the rays FH, FN, or any other rays which emerge respectively parallel to GI and GP. But (by Prop. CXXIV.) all the effectual violet rays are parallel to GI, and all the effectual red rays are parallel to GP. Therefore the effectual violet rays at their emerfion make a less angle with the incident rays than the effectual red rays. And universally the more refrangible

Plate 8.  
Fig. 2.



refrangible rays, at their emerſion, make a leſs angle with the incident rays than thoſe which are leſs refrangible. And ſince the effectual rays GI, GP, of different colours make different angles with the incident ray SK at their emerſion, they will be ſeparated from one another: ſo that if the eye was placed in the beam FGHI, it would receive only rays of one colour from the drop XY, and in FGNP only rays of another colour.

SCHOL. The angle which the effectual red rays make with the incident rays is found to be  $42^{\circ}. 20'$ . that of the violet rays  $40^{\circ}. 17'$ .

EXP. Let a glaſs globe filled with water be expoſed to the rays of the ſun: let the eye of the ſpectator be ſo ſituated, that the leaſt refracted ray from the drop, coming to the eye, ſhall make an angle of about  $42^{\circ}$ . with the line paſſing through the eye and the ſun, the red rays only will be ſeen: if the place of the eye be changed, ſo as to enlarge this angle, the red will diſappear; but if the angle be leſſened, the colours of the more refrangible rays will appear.

#### P R O P. CXXVII.

If a line is ſuppoſed to be drawn from the center of the ſun through the eye of the ſpectator, the angle which any effectual ray after two refractions and one reflection makes with the incident ray, will be equal to the angle which it makes with that line.

Plate 8.  
Fig. 2.

Let I be the place of the eye of the ſpectator; QT a line drawn from the center of the ſun through the eye; and AB a ray coming from the center of the ſun. Theſe two lines AB, QT, on account of the great diſtance of the ſun, may be looked upon as parallel. Therefore (El. I. 29.) the alternate angles AKI, KIT, or GIT, are equal.

#### P R O P. CXXVIII.

When the ſun ſhines upon the drops of rain as they are falling, the rays which come from thoſe drops to the eye of the ſpectator, after one reflection and two refractions, produce the innermoſt or primary rainbow.

Plate 8.  
Fig. 3.

Let TFY be the innermoſt or primary rainbow, the outer part of which TFY is red, the inner part VDX violet, and the intermediate parts reckoning from the red to the violet, orange, yellow, green, blue, indigo. Suppoſe the ſpectator's eye at A; and let AT be an imaginary line from the center of the ſun to the eye of the ſpectator. If a beam of light S coming from the ſun falls upon any drop F, and the effectual rays which emerge  
at



at F make an angle FAI of  $42^{\circ}. 2'$ . with the line AI, these rays (by Prop. CXXVII.) make the same angle with the incident rays, and consequently are red. Hence the drop F will appear red; for all the other rays which emerge from F, and would be effectual if they fell upon the eye, being refracted more than the red rays, will pass above the eye. If another beam of light S falls upon the drop D, and the effectual rays emerging at H make an angle of  $40^{\circ}. 17'$  with the incident rays, the drop D will be of a violet colour: for all the other rays which emerge from H, and would be effectual if they came to the eye, being refracted less than the violet rays, will pass below the eye. The intermediate drops between F and D will for the same reasons be of the intermediate colours. And that which has been proved concerning the drops in the line FD, may be shewn of any other set of drops in which the angles made by the emerging and incident rays are equal. Thus, wherever a drop of rain is placed, if the angle which the effectual rays make with AI is equal to the angle FAI, or is  $42^{\circ}. 2'$ . any such drop will appear red. If FAI was turned round upon the line AI, so that one end of this line should always be at the eye, and the other at I opposite to the sun, in this revolution the drop F would describe a circle, of which I would be the center, and TFY an arc. And since in this revolution the angle FAI continues the same, if the sun was to shine upon this drop as it revolves, the effectual rays (by Prop. CXXVII.) would make the same angle with the incident rays in whatever part of the arc TFY the drop may happen to be; and consequently in whatever part of the arc the drop F is, it will appear red. Now as innumerable drops are falling at once in right lines from the cloud, whilst one drop is at F, there will be others at T, Y, and every other part of the arc, which will appear red in the same manner that F would have done in the supposed circular revolution. Therefore when the sun shines upon the rain, there will be a red arc AFB produced opposite to the sun. In like manner a violet arc VDX will be produced, and other intermediate arcs of the several intermediate colours, which will together make up the primary rainbow.

## P R O P. CXXIX.

The primary rainbow is never a greater arc than a semicircle.

Since the line AI is drawn from the sun through the eye of the spectator, and through I the center of the rainbow, this center is always opposite to the sun. And since the angle FAI is an angle of  $42^{\circ}. 2'$ . the highest part of the bow is  $42^{\circ}. 2'$ . from I its center. If therefore the sun is more than  $42^{\circ}. 2'$ . above the horizon, I, which is opposite to it, must be more than  $42^{\circ}. 2'$ . below the horizon, and no primary rainbow will be seen. As much as the altitude of the sun is less than  $42^{\circ}. 2'$ . so much will the highest point F of the rainbow be above the horizon: and when the sun is in the horizon, I the center of the bow will also be in the horizon on the opposite side, and half the circle will be visible; but when the sun is set, no bow can be seen.

Plate 8.  
Fig. 3.

E e

P R O P.



## P R O P. CXXX.

When the rays of the sun fall upon a drop of rain, some of them after two reflections and two refractions may come to the eye of a spectator who has his back towards the sun and his face towards the drop.

Plate 8.  
Fig. 5.

If parallel rays from the sun, ZV, YW, fall upon the lower part of the drop of rain BGW, they will be refracted towards the perpendiculars VL, WL, in entering the drop, and proceed in the direction VH, WI. At HI some part of these rays will (by Prop. XLIII.) be reflected into the directions HF, IG. And some of these rays will be again reflected at F, G, into the directions FD, GB; which rays, when they emerge out of the drop at B and D, will be refracted from the perpendiculars, and may come to the eye of a spectator whose back is towards the sun and his face towards the drop.

## P R O P. CXXXI.

Those rays which are parallel to one another after they have been once refracted and once reflected in a drop of rain, will be effectual when they emerge after two refractions and two reflections.

Plate 8.  
Fig. 5.

The contiguous rays ZV, YW, being refracted towards the perpendiculars VL, WL, when they enter the drop, will (by Prop. XVIII.) become convergent; and because these rays fall upon the drop very obliquely, their focus will not be far from the surface VW. If this focus is at K, the rays after they have passed the focus, will diverge from thence in the directions KH, KI: and, if KI is the focal distance of the concave reflecting surface HI, the reflected rays HF, IG, (by Prop. L.) will be parallel. These rays are reflected again from the concave surface FG, and will meet in a focus at E, so that GE will be the focal distance of this reflecting surface: and because HI, FG, are parts of the same sphere, the focal distances GE, KI, are equal. When the rays have passed the focus E, they will diverge in the lines EB, ED. Now, if the rays VK, WK, when they have met at K, were to be turned back in the directions KV, KW, on emerging at V and W, they would (by Prop. XX.) be refracted into the lines of incidence, and become parallel. But since GE is equal to IK, the rays ED, EB, which diverge from E, fall in the same manner upon the drop at D and B, as the rays KV, KW, would fall upon it at V and W, and ED, EB, have the same inclination to the refracting surface DB, as KV, KW, would have to VW: whence the rays ED, EB, emerging at D and B, will be refracted in the same manner, and will have the same situation with respect to one another, as KV, KW, would have, that is, will



will be parallel to one another; having been contiguous before their entrance into the drop, they will therefore (by Prop. CXXIV.) be effectual.

## P R O P. CXXXII.

When effectual rays emerge from a drop of rain after two reflections and two refractions, those which are most refrangible will at their emergence make a greater angle with the incident rays than the least refrangible will make with them; by which means the rays of different colours will be separated.

Let BM, BA, a violet and a red ray, emerge from B; the angle which the violet ray BM makes with the incident ray YW is  $YrM$ ; and that which the red ray BA makes with the same is YSA. And since BSY, the external angle of the triangle BrS, is (El. I. 16.) greater than the internal angle BrS or BrY;  $YrM$ , the complement of BrS, is greater than YSA the complement of BSY. Consequently, since the emerging rays make different angles with the same incident ray, the refraction which they suffer at emergence will separate them from one another.

Plate 8.  
Fig. 5.

SCHOL. The angle which the violet rays make with the incident ones is found to be  $54^{\circ}. 7'$ . and that of the red rays  $50^{\circ}. 57'$ .

## P R O P. CXXXIII.

If a line is supposed to be drawn from the center of the sun through the eye of the spectator, the angle which, after two refractions and two reflections, any effectual ray makes with the incident ray, will be equal to the angle which it makes with that line.

If YW be an incident ray, and BA an effectual ray, and AO a line drawn from the center of the sun through A the eye of the spectator, YW and AO may be considered as parallel; whence the alternate angles YSA, SAO, (El. I. 29.) will be equal.

Plate 8.  
Fig. 5.

## P R O P. CXXXIV.

When the sun shines upon the drops of rain as they are falling, the rays which come from those drops to the eye of the spectator after two reflections and two refractions, produce the outermost or secondary rainbow.



Plate 8.  
Fig. 4.

When the sun shines upon a drop of rain E in the outer edge of the secondary rainbow CBD, the effectual violet ray EA (by Prop. CXXXII. Schol.) makes an angle EAI of  $54^{\circ}. 7'.$  with AI a line drawn from the sun through the eye of the spectator, and therefore (by Prop. CXXXIII.) make the same angle with the incident ray SB. Therefore if the spectator's eye is at A, all the rays except the violet, will (by Prop. X.) make a less angle with AI than EA, and fall above the spectator's eye. In like manner it may be shewn, that from the drop F only red rays will come to the spectator's eye, the rest falling below it; and that the rays emerging from the intermediate drops between E and F, and coming to A, will emerge at intermediate angles, and present to the eye the intermediate colours. If EAI be conceived to turn round upon the line AI, in such a revolution of the drop E, the angle EAI would remain the same, and consequently the emerging rays would make the same angle with the incident rays. But in such a revolution the drop E would describe a circle, of which I would be the center, and CBD an arc. Consequently, since the emerging rays make the same angle with the incident ones when the drop is at any other part of the arc as at E, the colour of the drop will be violet to an eye placed at A, in whatever part of the arc the drop is placed. Now, since there are innumerable drops of rain falling at once, whilst one drop is at E, there will be others in all parts of the arc, which will all appear violet-coloured, for the same reason that E would have appeared of this colour in any other part of the arc. In like manner, as the drop F appears red at F, and at any part of the arc FD, so will any other falling drop when it comes to any part of that arc. The intermediate arcs are formed in the same manner with the violet arc CBD, and the red arc FD; and thus the whole secondary rainbow is produced.

#### P R O P. CXXXV.

The colours of the secondary rainbow are fainter than those of the primary, and are ranged in the contrary order.

At every reflection many rays pass out of the drop without being reflected; consequently, the secondary rainbow which is produced after two reflections, is formed by fewer rays than the first, which is produced after one reflection.

Plate 8.  
Fig. 3.

Again, in the primary bow, the violet rays, when they emerge effectually, make a less angle with the incident rays (by Prop. CXXVI.) and therefore (by Prop. CXXVII.) with the line AI, than the red rays. But the rays are here only once reflected, and the angle which the effectual rays make with AI is the distance of the coloured drop from I the center of the bow. Therefore the violet arc in the primary bow will be nearer to the center of the bow than the red arc, that is, the innermost colour will be violet, and the outermost red. But in the secondary rainbow, the rays are twice reflected; and (by Prop.



Prop. CXXXII.) the violet rays, which emerge so as to be effectual after two reflections, make a greater angle with the incident rays, that is, with the line AI, than the red ones; which angle is the distance of the violet arc from I the center of the bow. Therefore the violet arc in the secondary bow will be farther from the center of the bow than the red arc; that is, the outermost colour is violet, and the innermost red.

## P R O P. CXXXVI.

The secondary rainbow is never a greater arc than a semicircle.

This is proved in the same manner as Prop. CXXIX. with this difference, that, since the rays of the highest colour in the secondary bow make an angle of  $54^{\circ}. 7'$ . with AI, this bow will begin to appear when the altitude of the sun is less than  $54^{\circ}. 7'$ . and when the sun is in the horizon on one side, this bow will have its center in the horizon on the other side at the distance of  $54^{\circ}. 7'$ . from its highest point.



## C H A P. VI.

## O F O P T I C A L I N S T R U M E N T S.

## S E C T. I.

*Of* T E L E S C O P E S.

DEF. XXX. An *Astronomical Telescope* consists of two convex lenses, whose distance from each other is equal to the sum of their principal foci: that lens which is towards the object, is called the *object-glass*; that which is next the eye, is called the *eye-glass*.

Plate 8.  
Fig. 7.

If NL is one convex lens, whose focal distance is MF, and BD another, whose focal distance is CF; and if these are so placed that the distance between them is equal to MF added to CF, that is, MC, they form an astronomical telescope.

## P R O P. CXXXVII.

Very remote objects, seen through an astronomical telescope, appear distinct and inverted.

Plate 8.  
Fig. 7.

Let PM, PL, be rays coming (by Prop. VIII.) parallel from the middle point in a very distant object: let AN, AM, AL, come from the highest point, and QN, QM, QL, come from the lowest point. These parallel rays will (by Def. XVIII.) be collected into the focus, and there form an image of the object, which (by Prop. LXXXII.) forms the object of refracted vision. But, by the construction of the telescope, GFE is the focus of the eye-glass. Consequently, the rays which diverge from any point G in this image will (by Prop. XX.) after they have passed through the eye-glass, become parallel. Therefore if the eye is at any point on the other side of the eye-glass, the object of refracted vision may be seen as distinctly as any very remote object can be seen by the naked eye; and because the image is the object of vision (by Prop. XXV.) it will be seen inverted.

P R O P.



## P R O P. CXXXVIII.

The apparent diameter of an object seen through an astronomical telescope, is to the apparent diameter of the same object seen by the naked eye at the station of the object-glass, as the distance of the image from the object-glass is to its distance from the eye-glass.

If the image, formed by the object-glass NL, were received upon a paper at EFG, the apparent diameter of the object seen by the naked eye at M the station of the object-glass, would be (by Prop. LXXXIV.) equal to the apparent diameter of the image seen from the same station. Now the real diameter of the image is given, because its distance MF from the lens is given. Consequently, the apparent diameter of the image (by Prop. LXIX.) will be inversely as the distance of the eye from it. If the eye is placed at C the station of the eye-glass, and consequently its distance from the image is FC, the image will appear to the eye in that station bigger than at the station M (by Prop. LXXXIX.) in the inverse ratio of the distances FC, MF, that is, the apparent magnitude of the image at C will be to that at M, as MF to FC. But the apparent magnitude of the image seen from M is equal to that of the object seen by the naked eye. Therefore the image seen from C appears bigger than the object, in the ratio of MF to FC. This would still be the case (by Prop. LXXXIV.) if the eye-glass were placed between the eye and the image, touching the eye. And since the image is in the focus of the eye-glass, the apparent magnitude (by Prop. XC.) is the same, whether the eye is close to the lens, or at any distance from it. Therefore wherever the eye is, the apparent diameter of the object seen with the telescope, is to the apparent diameter of the same object seen by the naked eye at the station of the object-glass, as MF to FC, or as the distance of the distinct image from the object-glass, to its distance from the eye-glass.

Plate 8.  
Fig. 7.

## P R O P. CXXXIX.

A telescope will not magnify an object, unless the focal distance of the object-glass is greater than the focal distance of the eye-glass.

The rays which come from distant objects being nearly parallel, the image GFE (by Def. XVIII.) will be in the focus of the object-glass, which, by the construction of the telescope, is also the focus of the eye-glass. But the apparent diameter of an object seen through a telescope, is to its apparent diameter when seen by the naked eye (by Prop. CXXXVIII.) as the distance of the image from the object-glass, to its distance from the eye-

Plate 8.  
Fig. 7.



eye-glass; that is, by what has been just proved, as the focal distance of the object-glass, to the focal distance of the eye-glass. Consequently, if MF the focal distance of the object-glass, is greater than FC the focal distance of the eye-glass, the object will be magnified: if MF be equal to FC, the object will appear as to the naked eye; if MF be less than FC, the object will appear diminished.

COR. 1. Hence the object-glass of a telescope should be less convex than the eye-glass.

COR. 2. An object will be equally magnified by two telescopes of very different lengths, if the ratio of the focal distances of the object-glass and eye-glass be the same in each.

COR. 3. If a telescope is inverted, objects seen through it will be diminished: for the object-glass which has the greater focal distance then becomes the eye-glass.

#### P R O P. CXL.

The visible area, or space which may be seen at one view through a telescope, is as the area of the eye-glass.

Plate 8.  
Fig. 7.

If GFE is any image, its distance from the object-glass being equal to the focal distance of the lens, the area of the image (by Prop. XXXVI.) is given; but the quantity of this image which can be seen at one view must be greater or less according to the magnitude of the hole through which it is seen, that is, must be as the area of the eye-glass.

#### P R O P. CXLI.

The brightness of an object seen through a telescope depends upon the area of the object-glass, but not the visible area.

The brightness of the image, that is, of the object of refracted vision, is (by Prop. XXXVIII.) as the area of the lens which forms it, that is, of the object-glass. But (by Prop. XXXVI. Schol. 2.) the magnitude of the image is the same, whether the area of the object is great or small; and consequently, if we look at it through an eye-glass of a given area, the quantity to be seen at once will not be altered by any change in the area of the object-glass.

#### P R O P. CXLII.

The distance of the eye from the eye-glass, should be equal to the principal focal distance of the eye-glass.

Since



Since the image GFE is in the focus of the lens DCB, wherever the eye is placed on the other side of the glass, the image will appear equally magnified. But when the eye is just as far from the eye-glass as its focal distance, the visible area will be the greatest: for, in that case (by Def. XVIII.) none but rays parallel, before the refraction, to MC the axis of the telescope, and therefore to the sides of the cylindrical tube in which the lenses are placed, can reach the eye, and consequently, no rays can come from the inner surface of this tube to the eye to make it visible: whereas in any other station of the eye, oblique rays from that surface would make the sides of the tube visible; whence the area of vision, which remains the same, being (by Prop. CXL.) always as the area of the eye-glass, will be in part occupied by the sides of the tube, and the object will be seen only through the remaining part.

Plate 8.  
Fig. 7.

DEF. XXXI. A telescope consisting of four convex lenses is a *double Astronomical Telescope*.

Let the two lenses NML, and B, placed at the distance MB, equal to the sum of their focal distances, form one telescope, and the two lenses C, D, placed at the distance CD, equal to the sum of their focal distances, form another. If these two telescopes are fixed at the distance CB from each other, so as to be both used together, they form a double telescope: the lens next to the object LMN is called the object-glass, and the lens B next the object-glass is called the first eye-glass, C the second, and D next to the eye the third.

Plate 8.  
Fig. 8.

### P R O P. CXLIII.

An object seen through a double telescope appears distinct and erect.

The parallel rays which fall upon the object-glass NML (by Prop. CXXXVII.) form a distinct inverted image at GFE the focus of the object-glass. This image being also in the focus of the first eye-glass B, the rays of each beam from the several points of this image will become parallel by passing through B: whence, falling parallel on the second eye-glass C, they will form a distinct inverted image at KIH the focus of this second eye-glass: and because KIH is also the focus of the third eye-glass D, the rays from this image, after passing through this third eye-glass, will come to the eye parallel to each other. Consequently, the object will be seen distinctly: and because the second image is inverted with respect to the first, which is inverted with respect to the object, the second image, or object of refracted vision, is in the same situation as the object itself.

Plate 8.  
Fig. 8.

F f

P R O P.



## P R O P. CXLIV.

A double telescope magnifies an object in the ratio of the focal distance of the object-glass, to the focal distance of the first eye-glass.

Plate 8.  
Fig. 8.

The first telescope MB magnifies the object in the ratio of MF to FB: and the second telescope CD is commonly made up of two lenses of equal convexities, which will not alter the apparent magnitude of the objects. Therefore, when both are used together, the object is only magnified by the first in the ratio of MF the focal distance of the object-glass, to FB the focal distance of the first eye-glass.

SCHOL. 1. The different refrangibility of the rays of light makes refracting telescopes imperfect: for those rays which are most refracted by passing through the lens, will be brought to a focus, and form an image nearer to the object-glass than those which are less refracted; and consequently the several sorts of rays are not properly collected in one focus to produce a perfectly white image, but each has its own focus, producing a confused and coloured image.

Of two refracting telescopes which magnify equally, the shorter will give a more imperfect image than the longer. For the image appearing equal in both, but being farther from the object-glass in the longer than the shorter, must be in reality larger or more magnified: whence the defect arising from the different refrangibility of the rays will be more visible in the longer than in the shorter telescope. Hence, reflecting telescopes are more perfect than refracting ones; for when all the rays are reflected, their angles of incidence and reflection being equal, they will all meet in a focus at the same distance.

SCHOL. 2. To remedy the defect of refracting telescopes, arising from the different refrangibility of rays of light, a compound object-glass is made use of, consisting partly of white flint glass, and partly of crown glass, which have different refracting powers. These refract contrary ways; and the excess of refraction in the crown glass is made such, as to destroy the colour caused by the flint glass. A telescope thus formed is called achromatic.

## P R O P. CXLV.

*Several kinds of*

To explain the construction and use of ~~Reflecting~~ Telescopes.

## I. Of GALILEO's Telescope.

Galileo's telescope consists of a convex object-glass and a concave eye-glass, so placed that the distance between them is the difference of their focal distances.

In



In this telescope ZYX, a convex lens, is placed at the distance from BA a concave lens of YC, the difference between YF the focal distance of ZX, and CF the focal distance of BA.

Plate 8.  
Fig. 9.

From a distant object let rays fall upon the convex lens YZ, from which they will proceed towards the focus of this lens at FG. But the concave lens AB, the focus of which is at FG, renders the converging rays parallel when they reach the eye; whence an image will be formed upon the *retina*. And the pencils of rays being made more diverging by passing through the concave lens, the visible image is seen under a larger angle than the object, and appears magnified. Also, because the pencils which form the image only cross one another once, the image appears erect.

## II. Of SIR ISAAC NEWTON'S Telescope.

In a tube ABCD, towards the end BC, let the concave mirror GH be placed perpendicular to DC the lower side of the tube. If an object, which is at such a distance that rays coming from the same point may be considered as parallel to one another, be placed before the open end of the tube AD, these parallel rays will be reflected from the concave mirror GH, and becoming convergent, would (by Prop. CXII.) form an inverted picture of the object upon a paper held at the focus of the mirror. But if the converging rays, before they reach the focus, fall upon a plane mirror, K, placed at an angle of 45 degrees with DC the side of the tube, or with the axis of the telescope, they will be reflected from thence, and meet before it at L, forming an image perpendicular to the object, or parallel to the axis of the telescope. If this image be placed in the focus of a convex lens L, fixed in the side of the telescope, the eye will see it distinctly through the lens.

Plate 8.  
Fig. 10.

The image seen from the station of the eye-glass L, either with or without the glass, will (as in the refracting telescope, see Prop. CXXXVIII.) appear as much larger than when seen from the concave mirror, that is, as much larger than to the naked eye, as the distance of the image from the eye-glass, is less than its distance from the mirror, or as its distance from the mirror is greater than its distance from the lens.

## III. Of GREGORY'S Telescope.

In the tube TTYT let a concave mirror EN be placed. Any parallel rays OO, PP, from an object A, falling upon this mirror, will, after reflection, (by Prop. CXI.) form an inverted image at C its focus. Let C be more remote from a second smaller concave mirror PO (placed parallel and opposite to the first mirror EA in such manner that their axes shall be in the same straight line) than its focus. The rays which diverge from the several points of the image at C, and fall upon the mirror PO, will (by Prop. L.) converge after reflection; and consequently, if they pass through

Plate 8.  
Fig. 11.



a hole NM in the first mirror EA, they will form a second image which will be inverted in respect of the first, and in the same position with the object. If, whilst these rays are converging, they pass through a plano-convex lens *Spo* (placed in a smaller tube joined to the larger) they will be brought to a focus, sooner than they would otherwise have been, forming the second image F. This erect image is seen by the eye at O, through a meniscal eye-glass LL, whose convexity is greater than its concavity.

## S E C T. II.

## Of MICROSCOPES.

Plate 8.  
Fig. 6.

DEF. XXXII. A *single Microscope* is one convex lens placed between a small object and the eye.

DEF. XXXIII. A *double Microscope* consists of two convex lenses, of which the object-glass is more convex than the eye-glass; and the distance between them is equal to the distance of the image from the object-glass, added to the focal distance of the eye-glass.

Plate 8.  
Fig. 12.

Let AB, a convex lens, be the object-glass, and EF, another convex lens, be the eye-glass. Let the small object KL be ~~be~~ <sup>farther from</sup> the object glass than its focus; an image MDN of the object will (by Prop. XXIV.) be formed behind the glass; let the distance of this image from the object-glass be ID, and let its distance from the eye-glass be equal to the focal distance of the eye-glass: the distance of the two glasses from each other will be ID + DX, or IX, that is, the distance of the image from the object-glass, added to the focal distance of the eye-glass.

Some compound microscopes are made with three glasses, so that the rays after passing through AB the object-glass, and EF the eye-glass, are again made converging by a second eye-glass, and therefore brought sooner to a focus, than by the first.

COR. Hence it appears, that the difference between the microscope and telescope is, that in the telescope the rays of each pencil fall upon the object-glass nearly parallel, and are united in its focus; but in the microscope they fall upon it very much diverging from one another, and therefore form the image in a place beyond the focus, and consequently larger than the object.

## P R O P. CXLVI.

An object seen through a double microscope appears distinct and inverted.

The



The pencils of rays issuing from the objects KL, being transmitted through the object-lens AB, their foci will be in MN; where there will be an inverted image of the object, which is viewed through another lens, or eye-glass EF, the focus of which is at MN: hence a distinct and direct image is formed upon the *retina*, and it is seen inverted.

Plate 8.  
Fig. 12.

## P R O P. CXLVII.

The apparent diameter of an object seen through a double microscope is to that of the same object seen by the naked eye from the station of the object-glass, in the compound ratio of the distance of the image from the object-glass, to its distance from the eye-glass, and of the limit of distinct vision to the distance of the object-glass from the object.

The first part of this Proposition is demonstrated as Prop. CXXXVIII. And it is manifest, from Prop. LXIX. that if AB the distance of the object-glass from the object is less than the limit of distinct vision, the apparent diameter of the object will be as much greater than that of the object at the distance at which the naked eye can see it distinctly, as IC is less than that distance. Therefore the object is magnified, because the distance of the image from the object-glass is greater than its distance from the eye-glass, and also because the distance from the object is less than the limit of distinct vision. The magnifying power of the microscope is then in the ratio compounded of these two ratios.

## P R O P. CXLVIII.

When the same eye-glass is used, the magnifying power of the microscope will be increased by increasing the convexity of the object-glass.

For in order to keep the image in the focus of the eye-glass, when the convexity of the object-glass is increased, the object-glass must be brought nearer to the object: the consequence of which will be that the ratio of the limit of distinct vision to the distance of the object-glass from the object will (El. V. 8.) be increased: whence (by Prop. CXLVII.) the ratio of the apparent diameter of the object of refracted vision to that of the object seen by the naked eye will also be increased.

SCHOL. The aperture of the object-glass in a microscope must be small, else the outermost rays, diverging too much, will hinder the distinctness of vision: but, on account



account of the smallness of the aperture the object will appear faint, and it will be necessary, in order to remedy this, to illuminate the object as much as possible.

## P R O P. CXLIX.

To describe the construction and use of the Solar Microscope.

In a dark room, let a round hole be made in a window-shutter about three inches in diameter, through which the sun may cast a cylinder of rays into the room. In this hole let a tube be fixed, containing, a convex lens of about two inches in diameter, and three inches focal distance; the object, placed between two concave glasses, at the distance of about two inches and a half from the first convex lens; and a second convex lens, whose focal distance is a quarter of an inch, placed at this distance from the object. Let a plane mirror, connected with the tube, and moveable by means of a wheel, receive the sun's rays on the outside of the shutter, and convey them into the tube. The rays, passing through the first lens, will strongly illuminate the object, from which they will pass through the second lens, and form an inverted image of the object, magnified in the ratio of the distance of the object from the lens to that of the image from the lens.

## S E C T. III.

*Of the* MAGIC LANTERN.

## P R O P. CL.

To describe the construction and use of the Magic Lantern.

*Plate 9.  
Fig. 13.*

In the side of a lantern, let a tube be inserted, consisting of two parts, one moveable upon the other. In the moveable part let a convex lens GG be fixed; in the immoveable part let an object EE, painted with transparent colours upon a piece of thin glass, be placed; and in the fixed part of the tube, a convex lens, DD. This lens will cast a strong light from the candle upon the object EE. And when the rays which diverge from the several points of the object are, by the lens GG, made to converge, they will (by Prop. XXV.) form an inverted image of the object at KL, upon any white surface; provided that the object is farther from the lens than its focus, and that the whole apparatus is placed in a dark room. The image KL will be larger than the object



object EE, in proportion as the distance of the image from the lens is greater than the object. A concave reflector AB may be placed within the lantern, behind the candle, to increase the illumination of the picture EE. If the object be placed in an inverted position, its image will appear erect.

## S E C T. IV.

*Of the* CAMERA OBSCURA.

## P R O P. CLI.

To describe the construction and use of the Camera Obscura.

Let CD be a convex lens, and HK a plane mirror inclined at an angle of 45 degrees. An inverted image of the object AB would be formed at EG, where the foci of the rays from the object are found after refraction : but the rays being intercepted by the plane mirror HK, are reflected (by Prop. C.) to NM, the focal distance before it, making an angle with the mirror of 45 degrees ; whence the image will be in a position perpendicular to the object, at the top of the box, where, if the rays be received on a sheet of oiled paper, or a plate of glass unpolished on one side, it will be distinctly visible.

Plate 8.  
Fig. 14.

EXP. Let the several Optical Instruments, explained in the preceding Proposition, be exhibited, and their respective uses shewn.







# B O O K V.

## O F A S T R O N O M Y.

### P A R T I.

#### O F T H E M O T I O N S O F T H E H E A V E N L Y B O D I E S.

#### C H A P. I.

##### *Of the SOLAR SYSTEM in General.*

DEF. I. **T**HE *Solar System* consists of the Sun; six primary planets, Mercury, Venus, the Earth, Mars, Jupiter, Saturn; ten secondary planets, the Earth's Moon, Jupiter's four Satellites, and Saturn's five; and an uncertain number of comets.

SCHOL. In this system, according to the doctrine of Copernicus, the six primary planets move round their own axes, and also move round the sun, from west to east; and the secondary planets move round their respective primaries from west to east, at different distances, and in different periodical times.

According to this doctrine, the Sun *S* is the center of the system; Mercury *a*, Venus *b*, the Earth *t*, Mars *e*, Jupiter *f*, and Saturn *h*, revolve in elliptical orbits round the sun; the moon *d* revolves about the earth, and the satellites of Jupiter and Saturn revolve about their primaries; and the planes of their orbits are inclined to one another.

Plate 9.  
Fig. 1.

This doctrine, being admitted as true, will account for the apparent motions, and other phenomena, of the heavenly bodies, as will be seen in the following chapters.



## C H A P. II.

## O F T H E E A R T H.

## S E C T. I.

*Of the GLOBULAR FORM of the EARTH, and its DIURNAL MOTION about its AXIS, and of the Appearances which arise from these.*

## P R O P O S I T I O N I.

The earth is of a globular form.

For, 1. The shadow of the earth projected on the moon in an eclipse is always circular; which appearance could only be produced by a spherical body. 2. The convexity of the surface of the sea is visible; the mast of an approaching ship being seen before its hull. 3. The north pole becomes more elevated by travelling northward, in proportion to the space passed over. 4. Navigators have sailed round the earth.

EXP. Let the shadow of a circle and of a sphere fall on a plane surface.

DEF. II. The *Axis* of the earth is an imaginary line passing through the center, about which its diurnal revolution is performed.

DEF. III. The *Poles* of the earth are the extremities of this axis.

DEF. IV. The *Equator* is the circumference of an imaginary great circle passing through the center of the earth, perpendicular to the axis, and at equal distances from the poles.

EXP. The axis, poles, and equator of the earth, are represented on the terrestrial globe.

DEF. V. If the axis of the earth be produced both ways, as far as the concave surface of the heavens, in which all the heavenly bodies



bodies appear to be placed, it is then called the *Axis of the Heavens*; its extremities are called the *Poles of the Heavens*; and the circumference produced by extending the plane of the equator to the same concave surface, is called the *Equator in the Heavens*.

EXP. These are represented on the celestial globe.

DEF. VI. Circles drawn through the poles of the earth or heavens perpendicular to the plane of the equator, are called *Secondaries of the equator*.

EXP. Observe these represented on the globes.

DEF. VII. The *sensible Horizon* is an imaginary circle, which, touching the surface of the earth, separates the visible part of the heavens from the invisible. The *rational Horizon* is a circle parallel to the former, the plane of which passes through the center of the earth.

SCHOL. Since (by Book IV. Prop. LXIX.) the apparent diameter of an object is inversely as its distance, if the distance be increased in such manner that it may be looked upon as infinite, the apparent magnitude becomes a point. Hence AF, the semidiameter of the earth, viewed at the different distances  $o$ , O, R, diminishes, till at the distance of O, a fixed star, it becomes a point, and the star appears in the same place in the heavens, whether viewed from the visible horizon SET, or the rational horizon HBR.

Plate 9.  
Fig. 4.

DEF. VIII. The *Poles of the Horizon* are two points, the one of which, over the head of the spectator, is called the *Zenith*; the other, which is under his feet, is called the *Nadir*.

DEF. IX. Circles drawn through the zenith and nadir of any place, cutting the horizon at right angles, are called *Vertical Circles*.

DEF. X. A vertical circle passing through the poles of the heavens, is a *Meridian*, and is said to be the meridian of any place through which it passes.



DEF. XI. The meridian of any place passing through the poles, and falling perpendicularly upon the horizon, cuts it in two opposite cardinal points, called *North* and *South*.

DEF. XII. A *Meridian Line*, is the common intersection of the plane of the meridian and the plane of the horizon.

COR. Hence any line which lies due north and south in an horizontal plane may be considered as part of the meridian line.

SCHOL. I. To draw a meridian line, perpendicular to an horizontal plane, erect a wire seven or eight inches long, having its upper end broad, that the termination of the shadow may be distinctly perceived: mark, at several different times before noon, the points in which the shadow terminates; through these points draw concentric circles about the middle point of the wire's station; observe in the afternoon when the extremity of the shadow again touches these circles; and find the middle point of each arc between the points already taken: a line drawn through these middle points, and the common center, will be the meridian line; for, since at equal distances from noon the sun is at the same height, or in verticals equally distant from the meridian, the circle drawn through the zenith at equal distances from these verticals is the meridian.

*in this plane,* SCHOL. 2. To observe the transit of any heavenly body over the plane of the meridian; place a telescope, having two cross hairs before its object-glass, one vertical, the other horizontal, and observe when the vertical hair passes through the center of the heavenly body: or, hanging two plumb-lines exactly over the meridian line, place your eye close to one of the threads in such manner, as that it shall cover the other thread, and observe when the body is behind the threads.

DEF. XIII. The *Altitude* or *Depression* of any heavenly body above or below the horizon, is the arc of a vertical circle intercepted between the body and the horizon, or the angle at the center measured by that arc.

SCHOL. The altitude of any heavenly body is found by the help of a quadrant thus: bring the quadrant into such a situation that the star may be seen through the sights; then the angle, contained between the string of the plummet and the side of the quadrant on which the sights are not placed, is the altitude of the star.

DEF.



DEF. XIV. The *Prime Vertical*, is that which crosses the meridian at right angles in zenith and nadir, cutting the horizon in the cardinal points *East* and *West*.

DEF. XV. The *Azimuth* of a heavenly body, is the arc of the horizon intercepted between the meridian and a vertical circle passing through that body; it is eastern or western as the body is east or west of the meridian.

SCHOL. The azimuth of any star may be thus found. Let AC be a given meridian line. Above any point A in this line, let a cord with a plummet be hung: let another cord with a plummet be hung at E, so that the star and the two cords shall lie in one and the same right line. Let the perpendiculars AD, BE, represent the cords, and draw AB. From the point B, to any point C, in the meridian line AC, taken at pleasure, draw the right line BC; then with a scale of equal parts measure the three lines AB, AC, BC. In the triangle, therefore, ABC, there will be given all the sides, from whence will be found the angle BAC, equal to the azimuth required.

Plate 9.  
Fig. 5.

For if the meridian line be supposed to be continued to F, and the line BA to G, the angle FAG will be the azimuth of the star; but the angle FAG will be equal to the angle at the vertex BAC; therefore the angle BAC will be equal to the azimuth.

DEF. XVI. The *Amplitude* of a heavenly body at its rising, is the arc of the horizon intercepted between the point where the body rises, and the east; its amplitude at setting, is the arc of the horizon intercepted between the point where the body sets, and the west: it is northern, or southern, as the body rises, or sets, to the north or south of east or west.

DEF. XVII. If an heavenly body rises, or sets, when the sun rises, it is said to rise or set *cosmically*; if it rises, or sets, when the sun sets, it is said to rise or set *achronically*; it is said to set or rise *beliacally*, when it approaches so near the sun as to become invisible, or recedes so far from him as to become visible.

DEF.



DEF. XVIII. The *Latitude* of a place upon the surface of the earth, is its distance from the earth's equator; it is measured by the arc of the geographical meridian of the place intercepted between the place and the equator: latitude is either northern or southern.

DEF. XIX. *Parallels of Latitude*, are circles on the surface of the earth drawn parallel to the equator.

## P R O P. II.

A degree in the equator is to a degree in any parallel of latitude, as radius to the cosine of latitude.

Plate 9.  
Fig. 3.

Let EPQ be a geographical meridian, EQ the equator, and FB a parallel of latitude. The circumference EQ is to the circumference FB, and any part of EQ, to any similar part of FB, as CQ, or CB, the radius of EQ, to AB the radius of FB; and AB is the cosine of the arc BQ, which is the latitude of the parallel FB. Therefore a degree in EQ is to a degree in FB, as radius to the cosine of latitude.

DEF. XX. The *Longitude* of a place, is the distance between the meridian of that place, and the meridian of some other place, taken at pleasure, and called the first meridian; it is measured by the arc in the equator intercepted between these two meridians. Longitude is either eastern or western, and is measured 180 degrees each way.

## P R O P. III.

The altitude of one pole, and the depression of the other, at any place on the earth's surface, is equal to the latitude of that place.

Plate 9.  
Fig. 2.

Let R be a place upon the earth's surface; Z, N, its zenith and nadir; P, S, the poles of the heavens, and F, f, the poles of the earth; EE, the celestial equator, ee, the terrestrial, and HO the horizon. The latitude of the place is eR, or the equal arc EZ; and PO is the elevation of one pole, and HS the depression of the other. Because ZO is the distance of the zenith from the horizon, it is an arc of 90 degrees; and because EP is the distance of the pole from the equator, it is also an arc of 90 degrees:

ZO



ZO and EP are therefore equal. Take from each of these the common arc ZP, and the remainders EZ and PO are equal. But HS and PO are equal, because they subtend the equal angles HTS, PTO: therefore the elevation of one pole PO, and the depression of the other HS, are equal to the latitude of the place EZ.

COR. Hence the circumference of the earth may be measured, by measuring the length on the surface of the earth passed over in a line which lies north and south, while the pole gains one degree of elevation, and multiplying this length by 360. A degree of latitude contains  $69\frac{1}{5}$  English miles, whence 24930 miles is the measure of the circumference of the earth.

## P R O P. IV.

The elevation of the equator at any place is equal to the complement of its latitude.

Because ZO is equal to EP (each being an arc of 90 degrees) EZ is equal to PO, that is, (by Prop. III.) to the latitude of the place. But EH, the elevation of the equator, is the complement of EZ, it is therefore equal to the complement of the latitude of the place.

Plate 2.  
Fig. 2.

## P R O P. V.

The earth revolving daily round its axis from west to east, the heavenly bodies will appear to a spectator on the earth to revolve in the same time from east to west.

Let RCBF be the earth, T its center, HTO the rational horizon to a spectator at R whose zenith is Z; let a star appear in the horizon at H. The earth revolving from west to east, that is, in the order of the letters R, C, B, F, in a fourth part of one revolution, the spectator will be carried from R to C: consequently, his horizon will become ZN, and the star which appeared in his horizon at H when he was at R, will now appear nearly in the zenith. When another fourth part of the revolution is completed, the spectator will be at B, and N being now his zenith, and HO his horizon, the star will be set with respect to him, and will not rise till he is again in the station R, that is, till the earth has completed one revolution. Thus whilst the earth has turned once round upon its axis from west to east, all the heavenly bodies in the concave sphere of the heavens will appear to have turned round from east to west.

PROP.



## P R O P. VI.

The alternate succession of day and night is the effect of the revolution of the earth round its axis.

For, all the heavenly bodies appearing (Prop. V.) to move from east to west, while the earth revolves from west to east, the sun will appear, in each revolution, to rise above the horizon in the east, and after describing a portion of a circle, to set in the west, and will continue below the horizon, till by the revolution of the earth it again appears in the east: and thus day and night will be alternately produced.

SCHOL. The time of noon is found, by observing the instant when the center of the sun is cut by the perpendicular hair in a meridian telescope, as described Def. XII. Schol. 2. or by a sun-dial.

## S E C T. II.

*Of the ANNUAL MOTION of the EARTH round the SUN.*

## P R O P. VII.

The earth revolving round the sun in 365 days, 6 hours, 56 minutes, 4 seconds, the sun appears to revolve round the earth in the same time, but in the contrary direction.

Plate 9.  
Fig. 8.

Let S represent the sun, BAC the orbit of the earth, and FGHE the starry concave. Whilst the earth is moving from A through B to C, it is manifest that, to a spectator on the earth, the sun must appear to move in the contrary direction from E through F to G, in the great circle of the heavens formed by the plane of the earth's orbit. In like manner, while the earth is passing from C to A, the sun will appear to pass from G to E.

SCHOL. It is manifest that the circle in which the sun appears to move, is the same in which the earth would appear to move to a spectator in the sun. Hence the apparent place of the sun being found, the true place of the earth in its orbit is known.

DEF. XXI. The circle which the sun appears to describe annually in the concave sphere of the heavens, is called the *Ecliptic*.

DEF.



DEF. XXII. A portion of the heavens, about 16 degrees in breadth, through the middle of which passes the ecliptic, is called the *Zodiac*.

SCHOL. Within this zone lie the orbits of all the planets.

DEF. XXIII. The stars in the zodiac are divided into 12 *Signs*, Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius, Pisces. Figures, representing these signs, are drawn upon the celestial globe, in that portion of its spherical surface, which corresponds to the portion of the concave sphere of the heavens, in which the stars belonging to each sign are respectively placed.

Plate 9.  
Fig. 8.

### P R O P. VIII.

The axis of the earth in every part of the earth's revolution about the sun, makes, with the plane of its orbit, that is, of the ecliptic, an angle of  $66\frac{1}{2}$  degrees.

Let BA represent the plane of the ecliptic or earth's orbit, seen edgewise; S the sun; and Pp produced the axis of the equator. If the earth be at S, its axis is not perpendicular to the plane of the ecliptic, but makes an angle with it, PSA, about  $66^{\circ}.30'$ . In any other part of its orbit, as at M, or X, the axis of the earth PR is still inclined to the plane of the ecliptic in the same angle.

Plate 9.  
Fig. 7.

COR. I. The axis, in any one part of the orbit, is in a position parallel to that in which it was at any other part of the orbit. Supposing the line FG to represent the situation of the axis of the earth when at DFG, and to be parallel to the line HI; then when the earth is at *dfg*, or any other part of its orbit, its axis *fg* will still be parallel to the same line HI.

Plate 9.  
Fig. 6.

COR. 2. The planes of the equator and ecliptic, make with each other an angle of  $23\frac{1}{2}$  degrees.

DEF. XXIV. The ecliptic being divided into twelve equal parts, each of these parts is called a *Sign*; and the names of the signs in the ecliptic are the same with those in the zodiac, but do not exactly correspond with them.

H h

DEF.



DEF. XXV. The two points in which the ecliptic cuts the equator, are called the *Equinoctial Points*: the vernal equinox is at the first degree of Aries in the ecliptic; the autumnal, at the first of Libra.

SCHOL. The moment of time in which the sun enters the equator, may be found by observation, the latitude of the place of the observer being known. For in the equinoctial day, or near it, with an instrument exactly divided into degrees, minutes, and parts of minutes, take the meridian altitude of the sun: if it be equal to the altitude of the equator, or to the complement of the latitude, the sun is then in the equator; but if it is not equal, mark the difference, which will be the declination of the sun. The next day, again observe the meridian altitude of the sun, and gather from thence his declination. If these two declinations be of different kinds, as the one south and the other north, the equinox happens some time between the two observations; if they be both of the same sort, the sun has either not entered the equinoctial, or has past it. And from these two observations of the sun's declination, the moment of the equinox is thus investigated.

Plate 9.  
Fig. 9.

Let CAB be a portion of the ecliptic, EAQ an arc of the equator, and let their intersection be in A. Let CE be the declination of the sun at the time of the first observation, OD his declination in the second observation; the arc CO will be the motion of the sun in the ecliptic for one day. In the spherical triangle AEC, right-angled at E, we have the angle A which the equator and the ecliptic make, as also CE the declination of the sun, known by observation, by which may be found the arc CA. And in the same manner in the triangle AOD the side AO is found; and thence the arc CO, which is the sum or difference of the arcs CA, AO. Therefore as CO is to CA, so is 24 hours to the time between the first observation, and the moment of the ingress of the sun to the equinox.

DEF. XXVI. The points of the ecliptic which are at the greatest distance from the equator, are called the *Solstices*; and the circles which pass through these points parallel to the equator, are called the *Tropics*: the summer solstice is at the first of Cancer, the winter solstice at the first of Capricorn: the northern tropic is called the tropic of Cancer, the southern, of Capricorn.

COR. The sun is once in the year at each of the tropics, and twice at the equator.

DEF. XXVII. Circles which pass through the poles at right angles to the equator, or any other great circle, are called *Secondaries* to



to that circle: the secondary which passes through the equinoctial points, is called the *Equinoctial Colure*.

DEF. XXVIII. That pole which is nearest the tropic of Cancer, is called the *North Pole*, that which is nearest the tropic of Capricorn, is called the *South Pole*.

DEF. XXIX. An imaginary line passing through the center of the ecliptic, and perpendicular to the plane of it, is the *Axis of the Ecliptic*: its extremities are the *Poles of the Ecliptic*, and all circles, passing through these poles and perpendicular to the ecliptic, are its secondaries.

COR. The axis of the ecliptic makes an angle of  $23\frac{1}{2}$  degrees with that of the equator. Compare Prop. V. Cor. 2.

DEF. XXX. The *Polar Circles* are described by the revolution of the poles of the ecliptic about the poles of the equator; that which is next to the north pole, is called the *Arctic* circle; the opposite, the *Antarctic* circle.

DEF. XXXI. The *Declination* of any heavenly body, is its distance from the equator; this is either northern, or southern. The degrees of declination of any body are reckoned upon a secondary of the equator passing through that body.

DEF. XXXII. The *Right Ascension* of any heavenly body, is its distance from the first of Aries reckoned upon the equator: this is measured, by observing the arc which is intercepted between Aries and a secondary to the equator passing through the sun or star.

DEF. XXXIII. The *Latitude* of any heavenly body, is its distance from the ecliptic; and the degrees of latitude are reckoned on a secondary of the ecliptic passing through the body.

DEF. XXXIV. The *Longitude* of any heavenly body, is its distance from the first of Aries; and is measured on the ecliptic by the arc intercepted between the first of Aries and the secondary of the ecliptic which passes through the body: the longitude increases, as the body



recedes from Aries, through the whole revolution, till it reaches  $360^{\circ}$ . or comes again to Aries.

DEF. XXXV. Two bodies are said to be in *Conjunction* with each other, when they have the same longitude, or are in the same secondary of the ecliptic on the same side of the heavens, though their latitude be different: they are said to be in *Opposition*, when their longitudes differ half a circle, or they are on opposite sides of the heavens.

#### P R O P. IX.

The axis of the heavens is perpendicular to the planes of all the circles which the heavenly bodies describe in their apparent diurnal motions.

For the heavenly bodies, from the revolution of the earth round its axis, appear to move from east to west in circles perpendicular to the axis.

COR. 1. The planes of all these circles are parallel to the equator.

COR. 2. The axis passes through the centers of the circles.

DEF. XXXVI. The celestial sphere is called *right*, *oblique*, or *parallel*, as the celestial equator is at right angles, oblique, or parallel to the horizon.

#### P R O P. X.

In all places on the equator, the poles lie in the horizon, and all the circles of daily motion make right angles with the horizon.

For these places (by Def. XVIII.) having no latitude, the poles (by Prop. III.) are neither elevated above nor depressed below the horizon: and since the equator is  $90$  degrees from the poles, it is at right angles to the horizon, and also all circles parallel to it.

#### P R O P. XI.

Those who live at the equator are in a right sphere; and, consequently, their days and nights are always equal.

The



The great circle of the celestial equator and its parallels (by last Prop.) make right angles with the horizons of all places in the earth's equator; therefore (by Def. XXXVI.) the inhabitants of those places live in a right sphere. Hence, because the celestial axis PTS is in the plane of their horizon, and that this axis is at right angles to the plane of the equator, and (by Prop. X.) passes through its center and through that of all circles parallel to the equator, the plane of the horizon also passes through the centers of these circles; and consequently divides the equator and its parallels into two equal parts. One half of these circles will therefore always be above the horizon, and the other half below it. But each of the heavenly bodies in their daily motion describes some one of those circles, and the diurnal motion of the earth is uniform; therefore any heavenly body will, in this situation, be just as long above the horizon as below it. And because this will be the case with respect to the sun, as well as any other body, in whatever part of the heavens he is seen, the days and nights at the equator will always be of equal length.

Plate 9.  
Fig. 10.

## P R O P. XII.

At the poles of the earth, one celestial pole is in the zenith, and the other in the nadir; the equator coincides with the horizon, and all the circles of daily motion are parallel to the horizon.

For the latitude of the poles is 90 degrees from the equator, and the circles of daily motion are parallel to the equator.

## P R O P. XIII.

Those who live at either pole are in a parallel sphere; they see the heavenly bodies carried round them in circles parallel to the horizon; and their day, and their night, continues each half a year.

An inhabitant at P has the equator EQ in the horizon, and all its parallel circles also parallel to the horizon. Therefore each of the heavenly bodies, in its apparent daily motion, being in some one of these circles, must describe a path parallel to the horizon: so that those which are above the horizon will never set by this motion, and those which are below it will never rise. The sun therefore, in this situation, will not rise or set by the diurnal motion of the earth. But from the annual motion of the earth, the sun daily changes its apparent place in the heavens, till it has described the circle of the ecliptic CL; one half of which is above the horizon, and the other half below it, because these circles have a common center T, the center of the earth. Therefore, for one half of the year the sun will be in some part of CT, that half of the ecliptic which is above the horizon,

Plate 9.  
Fig. 10.



horizon, and will daily revolve in circles above the horizon; and for the other half, it will be in some part of TL, and will perform its daily revolutions in circles below the horizon.

## P R O P. XIV.

In any place between the poles and the equator, one celestial pole will be elevated, and the other depressed, at an angle less than a right angle; and the celestial equator will make an angle less than a right angle with the horizon.

For, since the place is not in the equator, it has some latitude; and since it is not at either of the poles, its latitude is less than 90 degrees: whence (by Prop. III.) the poles are elevated, or depressed, in an angle less than a right angle; and consequently the equator, which is perpendicular to the axis, makes an angle less than 90 degrees with the horizon.

## P R O P. XV.

Those who live on any part of the surface of the earth between the equator and either pole, are in an oblique sphere, and have all the circles of daily motion oblique to their horizon.

Plate 9.  
Fig. 10.

Let HO be the horizon of a place which lies between the earth's equator and either of its poles, the celestial equator EQ, and all its parallel circles, will be oblique to the horizon, and therefore each of the heavenly bodies, being in some one of these circles, will appear to move in a path oblique to the horizon.

## P R O P. XVI.

When the sun, in his annual apparent course, is in the points in which the ecliptic cuts the equator, the day and night will be of the same length at all places on the surface of the earth: but, when the sun is in any other part of the ecliptic, the days will be longer as the sun's declination towards the elevated pole increases, and shorter as its declination towards the depressed pole increases.

Plate 9.  
Fig. 10.

The plane of the horizon HO, of any place, passing through T, the center of the sphere, and also through the center of the equator, divides the equator CL into two equal



equal parts, one half above and the other half below the horizon. When therefore the sun has no declination, or is in the equator, it will appear in its daily revolution to describe the equator  $CL$ , and therefore, during one half of the revolution, it will be above the horizon, and, during the other half, below it.

But suppose the sun to have its declination towards  $P$ , the elevated pole, equal to  $Em$ : its diurnal apparent revolution will be in the circle  $mm$ , the center of which is in a part of the axis above the horizon; whence the plane of the horizon does not pass through the center, and consequently the circle  $mm$  is divided into two unequal parts, the greater above the horizon, and the less below it. Therefore the sun, describing the circle  $mm$ , with an uniform velocity, in its apparent diurnal revolution, will be longer in describing the part above the horizon than the part below it. And this difference manifestly increases, as the circle of the sun's apparent diurnal motion recedes from the equator, that is, as the sun's declination towards  $P$  increases. In like manner, it may be shewn, that the days will be shorter, as the sun's declination towards the depressed pole increases.

Or thus: Let  $AB$  represent the plane of the ecliptic seen edgewise;  $S$  the sun in the focus of the orbit;  $MO$ ,  $KL$ ,  $XY$ , the earth in different parts of its orbit. If  $FI$ , the axis of the ecliptic  $BA$ , were also the axis of the earth, that is, if the planes of the equator and ecliptic were coincident, it is manifest that the sun, the apparent annual motion of which is in the plane of the ecliptic, would at all times of the year appear to move in the circle of the equator, and to be equally distant from the poles, and consequently could produce, by its apparent motion, no varieties in the length of days and nights. But the earth's axis being inclined to the plane of its orbit, as  $Pp$ , when the earth is at  $MO$ , the pole  $P$  will be towards the sun, and the pole  $p$  turned from it, and the reverse when the earth is arrived at  $XY$ . When the earth is in the middle station between  $B$  and  $A$ , in either part of its orbit, both the poles will be in the circle illuminated as at  $KL$ .

In the position  $MO$ , since the sun must always illuminate one half of the globe, the light will pass beyond the pole  $P$  as far as  $F$ , and will extend towards the pole  $p$  no farther than  $I$ . Consequently, in the diurnal revolution of the earth round its axis, while the earth remains in this position, all the parts of the globe between  $F$  and  $G$  will be illuminated, and all the parts between  $I$  and  $H$  will be dark. Farther, in this position greater portions of those parallels which lie between the equator and the circle  $FG$ , will at any instant be in the illuminated, than in the dark, hemisphere; and, on the contrary, greater portions of those which lie between the circle  $HI$  and the equator, will at any instant be in the dark, than in the enlightened, hemisphere. Consequently, any given place on the side of the equator towards  $P$ , will, in one diurnal revolution, be longer in the light than in the dark, and the reverse on the side towards  $p$ . The difference between the length of day-light and night, will decrease on either side of the equator, as we approach towards it; and at the equator, the illuminated and dark portions of the circle being always equal, the days and nights will be of equal length. The contrary to all this will take place in the situation  $XY$ . Continual variations will take place, while the earth passes from  $MO$  to  $KL$ , and from  $KL$  to  $XY$ . But in the situation  $KL$ , the illumination  
extending

Plate 9.  
Fig. 7.



extending exactly to both poles, all the parallel circles are half illuminated, and half dark: consequently, any place upon the globe will, in a diurnal revolution, have equal portions of light and darkness; that is, day and night will be every where of equal length. This must happen twice in every annual revolution.

SCHOL. 1. When the sun is very near either of the tropics, the days do not appear of different lengths, for the circles of apparent diurnal motion are so near to each other, that they cannot be sensibly distinguished.

SCHOL. 2. The different degrees of heat at different seasons of the year are owing partly to the different lengths of the days, and partly to the different degrees of obliquity with which the rays fall upon the atmosphere at different altitudes of the sun.

### P R O P. XVII.

When the sun, or any other heavenly body, is in the equator, it rises in the east, and sets in the west.

For it then rises and sets in the points in which the equator cuts the horizon; that is, because the equator is at right angles to the meridian, which passes through the north and south points, in the points of east and west.

### P R O P. XVIII.

When the declination of the sun is towards the elevated pole, its meridian altitude is equal to its declination added to the elevation of the celestial equator: when its declination is towards the depressed pole, its meridian altitude is equal to its declination subtracted from the elevation of the equator.

Plate 9.  
Fig. 10.

Let HO be the horizon, T the earth, P and S the celestial poles, D the zenith, N the nadir, EQ the equator. If the sun be at C, having its declination towards the elevated pole P, when it arrives at the meridian PS, its meridian altitude CH is equal to the sum of CE its declination, and EH the elevation of the equator. If the sun be at I, having its declination towards the depressed pole S; when it arrives at the meridian, its altitude DH is equal to the difference of EH the elevation of the equator, and EI the sun's declination, as appears from the figure.

P R O P.



## P R O P. XIX.

When the declination of a heavenly body towards the elevated pole, is equal to the latitude of any place, the body will pass through the zenith of that place: and when its declination towards the depressed pole is equal to the latitude, it will pass through the nadir.

Any star or planet which passes through D, the zenith, in its apparent diurnal revolution must describe the circle DO; whence its distance from the equator or declination will be ED. But ED is the distance of the zenith from the equator, which, because the elevation of the equator is equal to the complement of latitude (Prop. IV.) is equal to the latitude. In like manner the reverse may be proved.

Plate 9.  
Fig. 10.

## P R O P. XX.

A heavenly body seen from any place, will never set from the diurnal motion of the earth, if the complement of its declination towards the elevated pole be equal to, or less than, the latitude of the place: and it will never rise, if the complement of its declination towards the depressed pole be equal to, or less than, the latitude.

Let PD, which is the complement of declination of a body at D, and also the distance of the body at D from the pole, be equal to PO, the elevation of the pole, or (by Prop. III.) the latitude; it is manifest, that the body at its lowest depression will be no farther from the pole than the horizon is, that is, will never be below it. In like manner the reverse may be shewn.

Plate 9.  
Fig. 10.

SCHOL. The latitude of a place may be found, by observing the greatest and least altitude of a fixed star that never sets.

Let A be a star near the north pole, which in its daily motion describes the circle AB without setting. A quadrant being placed in the plane of the meridian, or along the meridian line, observe its altitude when it is at A, and afterwards when at B; the difference of these altitudes is AB. And since the star, in its revolution about the pole, is always at equal distances from it, if AB be bisected in P, this point will be the pole, and consequently PO will be the elevation of the pole. But, since the lengths of the arcs AO, BO, have been found by observations, their difference AB, and the half of this difference, AP, or BP, is known: and PO is equal to  $BP + BO$ , or to  $AO - PA$ .

Plate 9.  
Fig. 10.

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Whence



Whence the elevation of the pole, that is, the latitude, is equal to the sum of the least altitude added to half the difference of the greatest and least altitude, or it is equal to the remainder arising from subtracting half the difference of the greatest and least altitudes from the greatest altitude.

Or, the latitude may be found from the sun's meridian altitude and declination. If the sun's meridian altitude, found by a quadrant, be CH, this altitude is equal to the sun's declination CE, added to the elevation of the equator EH. Therefore, if CE, the declination towards the elevated pole, be taken from the meridian altitude, the remainder EH will be the elevation of the equator. But since the elevation of the equator is the complement of latitude, the latitude is the complement of the elevation of the equator. This elevation therefore being found, the latitude of the place is known.

DEF. XXXVII. The two tropics and two polar circles upon the surface of the earth, divide it into five parts called *Zones*: the torrid zone lies between the two tropics; the temperate zones between the tropics and polar circles; and the frigid zones between the polar circles and the poles.

#### P R O P. XXI.

At any place in the torrid zone the sun is vertical twice every year.

The sun, in passing from the equator to the tropic of Cancer,  $23\frac{1}{2}$  degrees from the equator, has every northern declination from 0 to  $23\frac{1}{2}$ : and every place between the equator and the tropic of Cancer has some northern latitude between 0 and  $23\frac{1}{2}$ : therefore, in some part of its course from the equator to the tropic of Cancer, the sun must have a declination equal to the latitude of every place between the equator and the tropic: whence it must be once in the zenith of every such place in its course towards the tropic of Cancer. For the same reason it must be once in the zenith of every such place in its course from the tropic to the equator. The like may be shewn on the southern side of the equator.

#### P R O P. XXII.

The sun is vertical once every year at the places which lie in the tropics.

For the sun's declination is then  $23\frac{1}{2}$  degrees, equal to the latitude of the tropics.

P R O P.



## P R O P. XXIII.

At the polar circles, the longest day, and the longest night, is 24 hours.

When the sun is in the tropic of Cancer, the complement of its declination towards the elevated pole is  $66\frac{1}{2}$  degrees, equal to the latitude of the arctic polar circle: on this day therefore (by Prop. XX.) the sun will not set. When the sun is in the tropic of Capricorn, the complement of its declination towards the depressed pole will be  $66\frac{1}{2}$  degrees, equal to the latitude of the arctic pole; whence the sun will not rise during that day. The same may be shewn with respect to the antarctic circle.

## P R O P. XXIV.

The longest day, and the longest night, are each of them more than 24 hours within the frigid zone.

For, while the sun's complement of declination towards the elevated pole is less than the latitude of the place, the sun will not set; while the complement of declination towards the depressed pole is less than the latitude of the place, it will not rise: but this must be the case with respect to every place within the frigid zones, in some part of the sun's course towards the tropics.

## P R O P. XXV.

The sun is never vertical to any place in either of the temperate zones.

For the latitude of all places in the temperate zone, is greater than the complement of any declination of the sun.

## P R O P. XXVI.

The longest day, and the longest night, in any part of the temperate zones, are less than 24 hours; and the days and nights will be longer, the nearer the place is to the polar circles.

For the complement of the sun's declination can never be less than, or equal to, the latitude of any place in the temperate zones; whence the sun will rise and set every day within these zones. But the farther any place is removed from the equator, the nearer the latitude approaches to an equality with the complement of the sun's greatest declination, when the day is 24 hours; that is, at the polar circles.



## P R O P. XXVII.

At different places, the hour of the day differs in proportion to the difference of longitude; 15 degrees of longitude making the difference of one hour in time;  $15'$ , one minute of time,  $15''$ , one second of time: and it is seen at any given place sooner than at places which lie to the west of it, and later than at places which lie to the east of it.

The sun in its daily apparent motion, which is from east to west, must arrive at the meridian of any given place, as London, sooner than it will arrive at the meridian of any place which lies to the west of London, and later than at the meridian of any place to the east of London: that is, since it is noon at any place when the sun is in its meridian, it will be noon at London sooner than at places west, and later than at places east, of it.

For example, if any place lies 15 degrees east of London, that is, has 15 degrees of eastern longitude from London taken as the first meridian, the sun will be one hour sooner at its meridian than at the meridian of London; for, since the sun every day appears to make a complete revolution from any meridian to the same, in 24 hours, it will in every hour describe a 24th part of the circle, that is,  $15^\circ$ . And since a minute of a circle is a 60th part of a degree, and a second of a circle a 60th part of a minute, and  $15'$  the 60th part of  $15^\circ$ , and  $15''$  the 60th part of  $15'$ , the sun will move at the rate of  $15'$  in every 60th part of an hour, and  $15''$  in every 60th part of a minute, that is, in every minute or second of time. Consequently, it will be noon one minute or one second sooner at a place which is  $15'$  or  $15''$  east of London, than at London.

## P R O P. XXVIII.

The difference of longitude at two places may be found by observing, at the same time from both places, some instantaneous appearance in the heavens.

If the eclipse of Jupiter's innermost satellite, on the instant of its immersion into the shadow of Jupiter, be observed by two persons at different places, it will be seen by both at the same instant. But if this instant be half an hour, for example, sooner at one place, than at the other, because the places differ half an hour in their reckoning of time, their difference of longitude (by Prop. XXVII.) is  $7^\circ. 30'$ .

SCHOL.



SCHOL. From tables of eclipses correctly calculated for any place, the longitude of any place may be found by one observer. But such observations can only be made with certainty by land, on account of the motion of a ship at sea. In order to determine accurately the longitude at sea, it is necessary to have a clock which shall not be sensibly affected by difference of climate, difference of gravity at different places, or the motion of the ship. Such a clock set for the meridian of London would constantly shew the hour of the day at London, which it is easy to compare with the hour of the day where the ship is, found by observations on the sun or stars.

## P R O P. XXIX.

Those who live in opposite semicircles of the same meridian, but in the same circle of latitude, have opposite hours of the day, but the same seasons.

Being both on the same side of the equator and at the same distance from it, when the sun's declination makes it summer or winter in one of the places, it will be the same at the other : but because they are distant from each 180 degrees of longitude, when it is noon at one place it will be midnight at the other : these are called *Periæci*.

## P R O P. XXX.

Those who live in opposite circles of latitude, but in the same semicircle of the meridian, have opposite seasons of the year, but the same hour of the day.

When the sun has declination towards the north pole, it will be summer to those who live in the northern circle of latitude, and winter to those who live in the southern circle of latitude. But, having the same longitude, their hours of the day will be the same : these are called *Antæci*.

## P R O P. XXXI.

Those who live in opposite circles of latitude and opposite semicircles of the meridian, have both opposite seasons of the year, and opposite hours of the day.

Because they are in opposite latitudes, they will have opposite seasons ; and because they are in opposite semicircles of the meridian, they will have noon when it is midnight at the other : these are called *Antipodes*.

DEF.



DEF. XXXVIII. Twelve secondaries to the celestial equator being conceived to be drawn at equal distances from each other, that is, dividing the equator into 24 equal parts, and the meridian of any place being made one of these secondaries, they are called *Hour-Circles* of that place. Compare Prop. XXVII.

## P R O P. XXXII.

If the celestial sphere had an opaque axis, the shadow of the axis would always be opposite to the sun; and when the sun was on one side of any hour-circle, the shadow of the axis would fall upon the opposite side of the same hour-circle.

For all the hour-circles being secondaries to the equator, pass through the poles, and the celestial axis is in the plane of every hour-circle. And the shadow of any opaque body, being opposite to the sun, is in the same plane with the sun. Therefore in whatever hour-circle the sun is, the shadow of the supposed opaque axis would be in the plane of that circle and opposite to the sun, that is, while the sun is in one semicircle of any hour-circle, the shadow of the axis would fall upon the opposite semicircle.

COR. Hence as the sun performs its apparent course from east to west, the shadow of the supposed axis would move from west to east.

SCHOL. The gnomon of a sun-dial represents the supposed axis, and hence its shadow is a measure of time.

## P R O P. XXXIII.

The orbit in which the earth revolves about the sun, is elliptical.

It is known from observation, that the apparent motion of the sun, that is, the real motion of the earth, in the ecliptic, is not uniform. But by the universal law of bodies revolving about a center, if its orbit were circular, its velocity must be uniform; since (Book II. Prop. LXXII.) it must describe equal areas in equal times. Whereas if its orbit be an ellipse, and the sun be placed in one of the *foci*, the same law will require (see Book II. Prop. LXVIII.) that its velocity should not be uniform, but that in passing through its greatest distance C, to its least distance A, it should be accelerated, and in passing from the least distance A to the greatest C, it should be retarded. Since then the motion of the earth is in fact thus retarded and accelerated in different parts of its orbit, it is manifest, that its orbit is elliptical.

DEF.

*Plate 9.  
Fig. 8.*



DEF. XXXIX. The greatest distance of the earth or any other planet from the sun, is called its *Aphelion*; its nearest distance, its *Perihelion*; the longer axis of the ellipse is called the *Linea Apsidum*, the aphelion is also called the *Summa Apsis*, and perihelion the *Ima Ipsis*.

DEF. XL. The *Excentricity* of the earth, or any planet, is the distance between the sun and the center of the elliptical orbit.

## P R O P. XXXIV.

The sun is eight days longer in performing its apparent course through the six northern signs, than through the six southern signs.

Let ABCD be the orbit of the earth, S the sun, and EFGH the ecliptic. While the earth moves in its orbit from B through C to D, the sun appears to move in the ecliptic from H through G to F, passing through the six northern signs; and while the earth passes from D through A to B, the sun appears to move from F to H, through the six southern signs. Now the line HF bisects the circle EFGH, but divides the ellipse ABCD unequally. And, while the sun appears to pass through the northern signs, the earth passes through more than half of its orbit; and while the sun appears to pass through the southern signs, the earth passes through less than half of its orbit. Therefore, if the velocity of the motion of the earth were uniform, the sun must appear to be longer in passing through the six northern than the six southern signs. But whilst the earth is passing through the greater part of its orbit BCD, it is farther from the sun, and consequently moves slower than in the lesser part DAB. On both these accounts, the sun's apparent motion is slower in the northern signs than the southern: the difference is found by observation to be about eight days.

Plate 9.  
Fig. 8.

## P R O P. XXXV.

The apparent diameter of the sun is greater in winter than summer.

It is found by observation, that the diameter of the sun in winter is 32 minutes, 47 seconds; in summer, 31'. 40''.

COR. Hence it appears, that the earth, at the winter solstice, or Capricorn, is in its perihelion.

SCHOL.



SCHOL. 1. The difference between summer and winter in the degrees of heat, is owing chiefly to the different heights to which the sun rises above the horizon, and the different lengths of the days. When the sun rises highest, in summer, its rays fall less obliquely, and consequently more of them fall on the earth's surface than in winter; and when the days are long and the nights short, the earth and air are more heated in the day than they are cooled in the night, and the reverse.

SCHOL. 2. The doctrine of the Sphere having been explained in the preceding propositions, some of the more useful Problems to be performed on the Terrestrial and Celestial Globes are here subjoined.

PROBLEM I. *To find the latitude of any place.* Bring the place to the graduated side of the fixed brass meridian; the degree under which it is found, is its latitude. All places under the same degree are in the same latitude.

PROB. II. *To find the longitude of any place.* Bring the place to the fixed meridian; the distance of this meridian from the first meridian, measured on the equator, is the longitude of the place.

PROB. III. *To rectify either globe to the latitude of any place, the zenith, and the sun's place.* If the place be in the northern hemisphere, raise the north pole above the horizon; but if the place be in the southern hemisphere, raise the south pole. Then move the meridian up and down in the notches, till the degree of the place's latitude, counted upon the meridian, below the pole, cuts the horizon; and then the globe is adjusted to the latitude of the place.

Having elevated the globe according to the latitude of the place, count the same number of degrees upon the meridian, from the equator towards the elevated pole, and that point will be the zenith or vertex of the place. To this point of the meridian screw the quadrant of altitude, so that its graduated edge may be joined to the said point; then is the globe rectified for the zenith.

Bring the sun's place in the ecliptic to the meridian, and set the hour-index to 12 at noon; and then the globe will be rectified for the sun's place.

PROB. IV. *To determine the difference of time in different places.* Find the longitude of each place, and reduce the difference into time, allowing an hour for every 15 degrees, and proportionally for lesser parts; the difference of time will be found: if the place lies westward of another, it has its noon later than that other; if eastward, sooner.

PROB. V. *The latitude and longitude of any place being known, to find the place on the globe.* Bring the degree of the equator which expresses the given longitude to the fixed meridian, then find the given latitude on the meridian; under this point is the place sought.

PROB.



PROB. VI. *To find the distance between any two places, and their bearing, or relative situation with respect to the points of the compass.* Rectify the globe to the latitude of one of the places, and bring that place to the fixed meridian: then fix the quadrant of altitude to the uppermost point of the meridian, and putting its lower end between the horizon and the globe, slide it along, till it passes through the other place: the number of degrees on the quadrant between the two places, will give their distance, allowing  $69\frac{1}{2}$  English miles for each degree; and the number of degrees upon the horizon between the meridian and the quadrant, will give the bearing of the second place with respect to the first.

PROB. VII. *To find the right ascension and declination of the sun, or any star.* On the celestial globe find the day of the month under the ecliptic, against which is the sun's place, or find his place by an *ephemeris*; bring that point under the meridian, and the degree which is over the point is the sun's declination, and the degree of the equator then under the meridian will be the sun's right ascension. A star's declination and right ascension are found, by bringing the star on the globe to the meridian, and proceeding as with respect to the sun.

PROB. VIII. *To find what stars pass over, or near, the zenith of any place.* Having found the latitude of the place on the terrestrial globe, all those stars on the celestial globe, which pass under the same degree of the meridian with the given latitude, become vertical at that place.

PROB. IX. *To find what stars never rise, or never set, in any given place.* The globe being rectified for the given place, those stars which do not pass under the wooden horizon, never set; those which do not come above it, never rise.

PROB. X. *To represent the appearance of the heavens at any time.* Rectify the globe to the latitude; bring the sun's place in the ecliptic to the meridian, and set the horary index to the upper 12th hour: then turn the globe till the index points to the given hour.

PROB. XI. *The latitude of a place being given, to find the time of the sun's rising and setting, or any given day, at that place.* Having rectified the globe according to the latitude, bring the sun's place in the ecliptic to the graduated edge of the meridian, and set the horary index to the upper 12. Then turn the globe to bring the sun's place to the eastern part of the horizon, the index will point to the hour at which the sun rises; on the western side, to the time of its setting.

PROB. XII. *To find all the places on the globe to which the sun will be vertical on a given day.* Bring the sun's place to the fixed meridian, and observe the point of the sun's declination; all the places which, in turning the globe round, pass under that point, have the sun vertical on the given day.

PROB. XIII. *To find the sun's amplitude.* The globe being rectified for the latitude of the place, bring the sun's place to the eastern side of the horizon; the arc of the horizon intercepted between that point and the eastern point, is the sun's amplitude at rising.



PROB. XIV. *To find the sun's altitude.* The globe being rectified for the latitude, zenith, and sun's place, the number of degrees contained between the sun's place and the zenith, is the distance of the sun from the vertex at noon; the complement of which to 90 degrees, is the sun's altitude.

PROB. XV. *To find the place of any heavenly body upon the globe, its longitude and latitude being given.* Place the first degree of the quadrant of altitude, upon that degree of the ecliptic which expresses the given longitude, and the 90th degree on the pole of the ecliptic; the point of the globe which is under that degree of the quadrant which expresses the given latitude, is the place of the body: for the quadrant represents a secondary of the ecliptic, an arc of which between the body and the ecliptic is its latitude, and the arc of the ecliptic between the secondary and the first degree of Aries its longitude.

PROB. XVI. *To find the place of any heavenly body upon the globe, its right ascension and declination being given.* Bring that point of the equator which expresses the given right ascension to the meridian; the place sought is under that degree in the meridian, north or south, which expresses the given declination.

EXP. Let the Propositions in this Section be illustrated on the Globes, Armillary Sphere, or Orrery.

### S E C T. III.

#### Of TWILIGHT.

#### P R O P. XXXVI.

The atmosphere above the horizon is enlightened by the rays of the sun, when the sun itself is below the horizon.

Plate 9.  
Fig. II.

Let ADL be the surface of the earth; CBM, the surface of the atmosphere; A, any place upon the earth; PABN, the sensible horizon. When the sun is at G, any point below the horizon, it cannot be directly seen by a spectator at A. But, because rays from the sun at G can pass to the part of the atmosphere above the sensible horizon of the place A, this part of the atmosphere will be illuminated before the sun rises, or after it sets, and will become visible by reflection to the spectator at A; that is, twilight will be produced.

P R O P.



## P R O P. XXXVII.

When the evening twilight ends, or the morning twilight begins, a ray of the sun reflected from the highest part of the atmosphere describes, after reflection, a line which is in the plane of the sensible horizon.

As the sun is depressed, the extreme ray of light from the sun gradually recedes from C towards B, till at last it touches the horizon at B, from whence it is reflected in the direction BA, the plane of the horizon.

Plate 9.  
Fig. 11.

## P R O P. XXXVIII.

When the evening twilight ends, or the morning twilight begins, a line drawn from the sun to the top of the atmosphere is a tangent to the surface of the earth.

From E, the common center of the earth and atmosphere, draw a straight line to B, the point in the top of the atmosphere, from which (by last Prop.) the sun's ray is reflected to A when the evening twilight ends, or the morning twilight begins. Because this line EB is perpendicular to the reflecting surface of the atmosphere, ABE is the angle of reflection. And AB, the reflected ray, being in the plane of the sensible horizon, describes a tangent to the surface of the earth. At the same time, the sun must be at some point S, such that the incident ray SB shall make the angle of incidence SBE, equal to ABE the angle of reflection. Draw AE; and from E, the center to the incident ray SB, draw ED, making the angle DEB equal to the angle AEB. Hence (El. I. 26.) the side ED is equal to the side AE, that is, ED is a semidiameter of the earth; also, the angle EDB is equal to the angle EAB, or is a right angle: whence (El. III. 16. Cor.) SB is a tangent to the circle; that is, when the reflected ray describes a line in the plane of the sensible horizon, the line described by the incident ray falling upon the top of the atmosphere, is a tangent to the surface of the earth.

Plate 9.  
Fig. 11.

## P R O P. XXXIX.

If the time of the beginning of the morning, or end of the evening twilight is known, the height of the atmosphere may be determined.



It is found, that at the beginning of the morning, or the end of the evening twilight, the sun is  $18^{\circ}$ . degrees below the horizon. If therefore S is the place of the sun at that time, and ABN the horizon, the angle SBN is  $18^{\circ}$ . But at that time (by Prop. XXXVIII.) if B is the top of the atmosphere, the incident ray SB, and the reflected ray BA, touch the earth at D and A. And, because (El. III. 18.) in the quadrilateral figure ABDE, the angles ABD, AED, are together equal to two right angles, or ABD is the complement to two right angles of AED, and that ABD is also (El. I. 13.) the complement to two right angles of SBN, AED, the angle contained by the semidiameters AE, DE, is equal to SBN the angle of the sun's depression: whence AED, at the time supposed, is an angle of  $18^{\circ}$ . But, on account of the refraction of the sun's rays in coming into the atmosphere, his apparent place is, at the supposed time,  $30'$  higher than its true place. Allowance must therefore be made for this refraction in measuring the angle SBN, made by SB the line of direction in which the sun's ray appears to come, and BN a line in the plane of the sensible horizon. Consequently, the angle SBN, and its equal AED, is only  $17^{\circ}. 30'$ . and its half AEB is  $8^{\circ}. 45'$ .

Now, in the triangle AEB, HB, the height of the atmosphere, is the difference between HE the radius, and BE the secant. Therefore as HE is to HB, the difference between the radius and secant of an angle of  $8^{\circ}. 45'$ . so is the number of miles in HE, a semidiameter of the earth, to the number of miles in HB, the height of the atmosphere.

#### P R O P. XL.

The twilight is longest in a parallel sphere, and shortest in a right sphere; and, in an oblique sphere, the nearer the sphere approaches to parallel, the longer is the twilight.

In a parallel sphere, the twilight will continue till the sun's declination toward the depressed pole is  $18^{\circ}$ . but in this sphere his declination is never more than  $23\frac{1}{2}$  degrees; whence the twilight will only cease, whilst the sun's declination is increasing from  $18^{\circ}$ . to  $23\frac{1}{2}$  degrees, and decreasing again till in its decrease it becomes  $18^{\circ}$ . The twilight is here caused by the annual motion of the earth. In a right sphere, the sun appears to be carried, by the daily motion of the earth, in circles perpendicular to the horizon; whence it is carried directly downwards by the whole daily motion, and will arrive at  $18^{\circ}$ . below the horizon, the soonest possible: whereas, in an oblique sphere, its path is oblique to the plane of the horizon, and therefore will be longer before it is descended 18 degrees below the horizon: and the difference of the time of twilight will increase with the degree of obliquity. As the sun sets more obliquely at some parts of the year than others, the twilight varies in its duration.

#### S E C T.



## S E C T. IV.

*Of the EQUATION of TIME.*

## P R O P. XLI.

The time in which the sun completes one apparent diurnal revolution, is greater than that in which the earth revolves round its axis.

If the earth turns round its center *T* in the direction *RCB*, and at the beginning of one revolution the sun was seen at *Z*, in the meridian; from its apparent annual motion it will, after the diurnal revolution of the earth is completed, be seen advanced in its orbit towards *E*. The earth therefore must perform more than one revolution, and the spectator at *R*, after returning to the station from which he set out, must advance forwards to *e*, before the sun will be again in the meridian.

Plate 9.  
Fig. 2.

## P R O P. XLII.

The obliquity of the ecliptic to the equator would cause the daily increments of the sun's right ascension to be unequal, although the sun's motion in the ecliptic were uniform.

Let *E* be the first degree of Aries, *EQ* an arc of  $90^\circ$ . of the equator, *EC* the same of the ecliptic, and *CQ* an arc of the solstitial colure, between Cancer and the equator. At *E* the sun has neither longitude nor right ascension: these may therefore be considered as equal, when the sun sets out from Aries. At *C*, the longitude is equal to the right ascension; for both *EC* and *EQ* are by supposition  $90^\circ$ . degrees of great circles of the same sphere. But if the sun be any where between the first of Aries and the first of Cancer, as at *S*, the longitude will be *greater* than the right ascension *ER*. For *SR* being an arc of the secondary of the equator passing through the sun, *ES* is the longitude, and *ER* the right ascension: but *ES* is greater than *ER*, because the angle at *R* is a right angle, but the angle at *S* an acute angle. Now, if the sun be supposed to move uniformly in the ecliptic, or to describe equal arcs in equal times, the daily increments of longitude will be equal to one another; and consequently, since at the two extremes *E* and *C* the longitude and right ascension are equal, and the longitude is supposed to increase uniformly,

Plate 9.  
Fig. 13.

if



if the right ascension also increased uniformly, they would at all times be equal. But at S, R, or any other points in the same secondary, between the first of Aries and Cancer, the longitude is *greater* than the right ascension; the daily increments of right ascension are therefore unequal.

Plate 9.  
Fig. 14.

The longitude and right ascension are equal when the sun is at C and at E, the former being  $90^\circ$ . the latter  $180^\circ$ . from Aries both on the ecliptic and equator. But between C and E, the longitude is *less* than the right ascension; because ES, opposite to the right angle R, is greater than ER opposite to the acute angle S, and consequently the point S is nearer Aries than the point R. But, the sun being supposed to move uniformly, or to increase its longitude equally every day, if the right ascension also increased equally every day, since the longitude and right ascension are equal at C and E, they would be always equal. But at S, or any where between Cancer and Libra, the longitude is less than the right ascension: consequently, the daily increments of right ascension are not equal. In like manner it may be shewn, that in the third quarter the sun's longitude is greater than its right ascension, and in the fourth, less.

#### P R O P. XLIII.

If the plane of the ecliptic coincided with that of the equator, the daily increments of the sun's right ascension would nevertheless be unequal.

Because (by Prop. XXXIV.) the apparent annual motion of the sun is not uniform, it would in some days describe a longer arc than in others; that is, since its right ascension and longitude would in this case be the same, the daily increments of its right ascension would be unequal.

DEF. XLI. A *Natural Day* is the time the sun takes in passing from the meridian of any place, till it comes round to the same meridian again.

#### P R O P. XLIV.

Any place upon the earth's surface describes more than a circle round the earth's axis in a natural day; and the arc which it describes more than a circle in any day, is the sun's increment of right ascension for that day.

While



While the earth revolves round its axis, any place upon the earth's surface describes a circle: but (by Prop. XLIII.) while the sun completes its apparent diurnal revolution, any place on the earth's surface will move through one circle and an arc of a second; therefore any such place describes more than a circle (Def. XLI.) in a natural day.

And since both a meridian, and a secondary of the equator, passing through the poles, are perpendicular to the equator (Def. VI. and X.) if the sun at S be in SR the meridian of any given place, it is also in a secondary of the equator passing through that place. In like manner, if the sun be at T, and TV be after a natural day the situation of the meridian of the given place, the sun will be in TV, which will be both the meridian of the place, and a secondary of the equator. Whence, RV being part of the equator, since ER was the sun's right ascension when it was at S, and EV is its right ascension when it is arrived at T, RV must be the increment of the sun's right ascension for the natural day in which it is advanced from S to T. And, because SR, TV, are both perpendicular to the equator, and any place, in one diurnal revolution of the earth describes a circle parallel to the equator, RV taken in this circle will always be the same arc with RV in the equator, and therefore will be equal to the sun's daily increment of right ascension.

Plate 9.  
Fig. 13.

#### P R O P. XLV.

The natural days are not equal to one another.

For any natural day is the time in which the earth performs one revolution round its axis, and such a portion of a second as is equal to the sun's increment of right ascension for that day; but the sun's daily increments of right ascension are unequal (by Prop. XLII. and XLIII.); therefore the additional portion of the second revolution will sometimes be greater and sometimes less, and consequently the times in which the natural days are completed will be unequal.

COR. Hence arises the difference between a sun-dial and a clock, as measures of time: the former measuring the length of the natural day, the latter dividing time into equal portions, of 12 hours each, the clock will be before the dial, when the natural day is more than 24 hours (the mean length of the natural day) and after it, when the natural day is less than 24 hours; and they will be together, only when the natural day is exactly 24 hours.

DEF. XLII. The *Equation*, is the difference between the mean length of the natural day (or 24 hours) and the length of any single day measured by the sun's motion, or between *mean time* and *apparent time*.

COR.



COR. The hour of the day by apparent time being known, in order to determine what is then the true mean time, the equation is to be *added* to the apparent time, when the day by the clock is shorter than the day by the sun-dial, that is, when mean time *precedes* apparent time; and the equation is to be *subtracted* from the apparent time when the day by the clock is longer than the day by the sun-dial, that is, when mean time *follows* apparent time. If, for example, the natural day is 24 hours, 1 minute, the day by the clock being 24 hours in length, it will be 12 by the clock, one minute before it is 12 by the dial, or mean time precedes apparent time one minute; therefore it will be 1 minute past 12 by the clock, when it is exactly 12 by the sun; whence one minute, which is the equation, must be *added* to the apparent time, to give the true mean time, that is, the clock must be set at 1 minute past 12. The reverse of this, when mean time follows apparent time, or the day by the clock is longer than the day by the sun, is obvious.

## P R O P. XLVI.

If the sun were to move uniformly round the equator in the same time in which it appears to describe the ecliptic, its apparent daily motion would be a measure of mean time.

For the natural days in that case being liable to no variation, either from the declivity of the sun's orbit, or the irregularity of its motion, must be equal.

## P R O P. XLVII.

The portion of time which passes between the arrival of the sun in the ecliptic to the meridian of any place, and its supposed arrival at the same meridian, if it were to move uniformly in the equator, is the equation.

For (by the last Prop.) it would be noon by mean time at any place, when the sun, if it moved uniformly in the equator, was arrived at the meridian of that place; and it is noon at the same place by apparent time, when the sun in the ecliptic arrives at the same meridian: therefore the difference between these two arrivals, is the difference between mean time and apparent time, or the equation.

## P R O P. XLVIII.

In the time which passes between the arrival of the sun in the ecliptic to the meridian of any place, and its supposed arrival at  
the



the same meridian, if it were to move uniformly in the equator, an arc of the equator passes under the meridian, which is equal to the difference between the right ascension of the sun, as it moves in the ecliptic, and the right ascension which the sun would have, if it moved uniformly in the equator.

Let the sun be at S; and let EC be the ecliptic, EQ the equator, and E the first of Aries: then if a secondary of the equator passes through the sun, SR, being at right angles to EQ, is an arc of that secondary, and (by Def. XXXII.) ER is the sun's right ascension, and the point R is the point in which the right ascension ends; which being in the secondary of which SR is a part, that is, the secondary passing through the sun, arrives at the meridian at the same time with the sun. If the sun were to move uniformly in the equator, and were arrived at P, EP would be its right ascension, and consequently P would be the point in which its right ascension would end, which point P must arrive at the meridian at the same time with the sun, because the sun is supposed to be in that point. Therefore RP, the distance of the two points R and P, is an arc of the equator (passing under the meridian in the time specified in the Proposition) which is equal to the difference between the real and supposed right ascensions of the sun, when he arrives at the meridian by his real motion in the ecliptic, and when he arrives at the same meridian by an uniform motion in the equator.

Plate 9.  
Fig. 13.

### P R O P. XLIX.

The right ascension of the sun, if it were to move uniformly in the equator, would at any time be equal to the longitude which it would have at that time if it were to move uniformly in the ecliptic, or to its mean longitude.

For, on this supposition, the sun, describing the equator with an uniform velocity in the same time in which it actually describes the ecliptic, its velocity would be the same with the mean velocity in the ecliptic. Consequently, the distance of the sun from the first of Aries in the equator would at any time be the same with its distance from the same point in the ecliptic, if it were to move uniformly therein with its mean velocity: that is, its right ascension in the equator would always be equal to its mean longitude in the ecliptic.



## P R O P. L.

An arc of the equator, equal to the difference between the sun's right ascension and its middle longitude, at any given time and place, converted into time, is the equation.

It has been shewn (Prop. XLVIII.) that in the portion of time which passes between the arrival of the sun in the ecliptic to the meridian of any place, and its supposed arrival at the same meridian if it were to move uniformly in the equator, an arc of the equator passes under the meridian, which is equal to the difference of the right ascension of the sun as it moves in the ecliptic, and the right ascension which it would have if it moved uniformly in the equator. And it has been proved (Prop. XLVII.) that this portion of time is the equation, and (Prop. XLIX.) that the right ascension which the sun, at any given time and place, would have if it moved uniformly in the equator, is equal to its mean longitude in the ecliptic. Therefore, in the equation, an arc of the equator passes under the meridian, equal to the difference of the right ascension of the sun in the ecliptic, and its mean longitude. Consequently, if this arc be converted into time, that is, if for 15 degrees be taken an hour, for 15' one minute of time, for 15'' one second of time, the equation of time will be found.

## P R O P. LI.

If the sun's mean longitude be greater than its right ascension, mean time *follows* apparent time; if its mean longitude be less than its right ascension, mean time *precedes* apparent.

Plate 9.  
Fig. 13.

If the right ascension of the sun, as before supposed in the equator, EP, that is (by Prop. XLIX.) its mean longitude, be greater than the sun's real right ascension ER, the supposed place of the sun in the equator P, will be to the east of the point R, where the sun's real right ascension ends. Therefore when this point R, at apparent noon, is come to the meridian, the point P will not be arrived at the meridian; and mean noon will be later than apparent noon. Therefore, when the sun's middle longitude is greater than its right ascension, mean time *follows* apparent. In like manner the reverse may be proved.

COR. Hence in the *former* case the equation is to be *subtracted* from the apparent time found by the dial, and in the *latter*, to be *added* to it, in order to obtain the mean time.



## C H A P. III.

*Of the* INFERIOR PLANETS, MERCURY and VENUS.

DEF. XLIII. The *Elongation* of any planet is its apparent distance from the sun.

DEF. XLIV. The *Nodes* of the orbit of a planet, are the two points in which the orbit cuts the plane of the ecliptic: and a right line drawn from one node to the other, is the *Line of the Nodes*.

DEF. XLV. The *Limits* of the orbit of a planet, are two points in the middle between the two nodes.

DEF. XLVI. An inferior planet is in its *inferior* conjunction, when it is nearer the earth than the sun is, and in its *superior* conjunction, when it is farther than the sun is from the earth, and both in the same secondary.

Let A be the place of the earth in its orbit ABO, E the place of Venus in its orbit EHG, S the sun, and FD an arc of a circle in the heavens. Venus will be in its inferior conjunction when it is at E, and at its superior when it is at G, and both are in the same secondary.

Plate 9.  
Fig. 12.

## P R O P. LII.

An inferior planet is at its greatest elongation, when a line drawn from the earth through the planet, is a tangent to the orbit of the planet.

When the planet is at E, being in conjunction with the sun, it has no elongation. As it moves from E towards X its elongation increases, till at X, when A is a tangent to the orbit of Venus, its apparent place is F, and its elongation FQ, which is the greatest elongation it can have; for in passing from X to G its elongation decreases, till at G it becomes nothing. This will be true in elliptical as well as circular orbits.

Plate 9.  
Fig. 12.



## P R O P. LIII.

The inferior planets are never in opposition to the sun.

For in opposition the earth is between the sun and the planets, which can never happen when the orbit of the planet is nearer to the sun than that of the earth.

DEF. XLVII. A planet is in *Quadrature*, when it is 90 degrees in the celestial sphere distant from the sun.

## P R O P. LIV.

The inferior planets are never in quadrature.

Plate 9.  
Fig. 12.

The greatest angle of elongation is that contained by AQ, drawn from the earth through the sun, and AF a tangent to the orbit of the planet. Now if QAF were a right angle, AF would be (El. III. 18.) a tangent to the earth's orbit; but AF is a tangent to an orbit less than that of the earth; it therefore makes an angle with AQ less than a right angle, that is, QF the greatest elongation is less than 90 degrees.

COR. Hence the inferior planets never appear far from the sun.

## P R O P. LV.

While Venus is moving from the superior conjunction to the inferior, it sets after the sun; while it is moving from the inferior conjunction to the superior, it rises before the sun.

Plate 9.  
Fig. 12.

Whilst Venus is moving from G, its superior conjunction, through P, to E its inferior conjunction, being in the eastern part of its orbit, the sun will be westward of Venus; therefore Venus, if far enough from the sun, will be seen in the west after the sun is gone down, whence it is then called the evening star. But on the western side of its orbit, the sun being eastward of it, Venus will set before the sun, and consequently rise before it, whence it is then called the morning star.

## P R O P. LVI.

The greatest elongation of an inferior planet on one side of the sun is not always equal to that on the other.

For



For since the planet moves in an elliptical orbit, at the time of its greatest elongation on one side it may be in its aphelion; and at its greatest elongation on the other side, it may be in some part nearer the sun: hence its real distance from the sun at its elongations being unequal, its apparent distances will be so likewise.

## P R O P. LVII.

The apparent velocity of the inferior planets is greatest at the conjunctions.

Since the plane of the orbit of Venus is oblique to that of the earth, those parts of this orbit which are viewed by a spectator on the earth directly, would appear longer than other equal parts viewed obliquely; whence its motions, if uniform, must appear unequal. If the orbit EPGH of Venus be seen obliquely by an eye placed at A, the parts about E and G, or near the conjunctions, will be seen directly, for AE is perpendicular to a tangent at E; but the parts about X and P would be seen obliquely: whence the Proposition is manifest.

Plate 9.  
Fig. 12.

SCHOL. The time when an inferior planet will come again into a given situation with respect to the sun and the earth, may be thus found. Whilst Venus performs one revolution, the earth, whose periodical time is longer than that of Venus, will not have completed its revolution. Before Venus and the earth can be again in the inferior conjunction, Venus must therefore, besides its entire revolution, describe an arc equal to that which the earth has passed over: consequently, the number of degrees passed over by each, or their angular motions, in the same time, will be reciprocally as their periodical times; that is, as the periodical time of the earth is to the periodical time of Venus, so is the angular motion of Venus (which is equal to four right angles added to the angular motion of the earth between two inferior conjunctions) to the angular motion of the earth in the same time: whence (El. V. 17.) as the difference between the periodical times of the earth and Venus, is to the periodical time of Venus, so are four right angles, or  $360^\circ$ . to the number of degrees over which the earth passes in her orbit from one inferior conjunction to another.

DEF. XLVIII. The apparent motion of a planet, if seen from the earth, is called its *Geocentric Motion*; if seen from the sun, its *Heliocentric Motion*.

## P R O P. LVIII.

When the inferior planets are passing from their greatest elongation, through their superior conjunction, to their greatest elongation on the  
other



other side, their geocentric motion is direct, or they appear to move from west to east.

Plate 9.  
Fig. 12.

When Venus is at X, it appears to a spectator at A to be in the point F of the concave sphere of the heavens: when it has moved forwards in its orbit to H, G, N, P, it will appear successively in the several points P, Q, R, D. But the motion from F to D is from west to east; for whilst the sun and earth are on the same side of the planet, it must appear to move in the same direction, whether it be viewed from the earth or the sun, because the spectator at either station views the concave side of the planet's orbit: but from the sun it is always seen to move from west to east; therefore its apparent geocentric motion in this situation is direct, or *in consequentia*.

#### P R O P. LIX.

While the inferior planets are moving from the greatest elongation on one side, to the greatest elongation on the other, through their inferior conjunction, their geocentric motion is retrograde, or from east to west.

While the planet is in this situation, the convex side of its orbit is towards a spectator on the earth, but its concave side towards a spectator at the sun: hence the former will see the planet move in a direction contrary to that in which it will appear to the latter to move. Thus, when the planet is at P, it will appear in the heavens at D; and as it passes through E to X, it appears to move from D through R, Q, P, to F: but the motion from D to F is from east to west; therefore the apparent motion of the planet in this part of its orbit is retrograde, or *in antecedentia*.

#### P R O P. LX.

When the inferior planets are at their greatest elongation, they appear stationary.

At either of the greatest elongations P, of the planet Venus, AD is a tangent to the orbit; which so nearly coincides with a small arc of the curve, that a spectator's eye placed at the earth could not distinguish the tangent from this part of the curve: consequently, a planet, while it is in this part of the curve, will appear to lie in the tangent AD, that is, to be stationary in the point D.

SCHOL. In the preceding Propositions, the earth's motion in its orbit is not regarded, because the only effect of this motion on the appearances above described, is, that they take place in different parts of the heavens; whereas, without this motion, the inferior planets



planets would always be direct or retrograde in their motion, or be stationary, in the same parts of the heavens.

If ABO be the orbit of the earth, and EHG that of Venus; when the earth is at A, the sun's geocentric place is Q, and Venus, in order to be in conjunction, must be in the line AQ at E or O. If the earth stood still at A, and Venus were in her inferior conjunction at E, the superior conjunction would always be at O, and her greatest elongations at X and P. But if the earth be advanced in her orbit from A to *a*, the sun's geocentric place is now at *m*, and Venus, in order to be in conjunction, must now be in one of the points in which her orbit cuts the line *am*. Thus the places of the conjunctions, and consequently of the greatest elongations *af*, *ad*, are continually carried round the ecliptic in the order of the signs.

Plate 9.  
Fig. 12.

EXP. Illustrate the three preceding Propositions on the Orrery.

P R O P. LXI.

The heliocentric latitude of an inferior planet is the greatest when the planet is in one of its limits.

For the planet is then (Def. XLV.) at its greatest distance from the ecliptic, and therefore will have the greatest latitude, as seen from the sun.

P R O P. LXII.

The geocentric latitude of an inferior planet is directly as its heliocentric latitude, and inversely as its distance from the earth.

The apparent length of a line drawn from the planet to the plane of the ecliptic, that is, its geocentric latitude, is (by Book IV. Prop. LXIX, LXX.) directly as its real length, and inversely as the distance of the spectator's eye. But the real length of a line drawn from the planet to the plane of the ecliptic, is its heliocentric latitude; and the spectator's eye is at the earth: whence the Proposition is manifest.

COR. When Venus is in its inferior conjunction, its heliocentric latitude is less than its geocentric; for it is then farther from the sun than from the earth. The contrary takes place with respect to Mercury.

P R O P. LXIII.

The sun enlightens only one half of a planet, and only one half of a planet is visible, at once.

This is sufficiently manifest from the spherical form of the planets, and the rectilinear motion of light.

DEF.



DEF. XLIX. The hemisphere of a planet which is towards the earth is called its *Disc*, because it appears like a plane circle.

## P R O P. LXIV.

The inferior planets are invisible in their inferior conjunction; their whole disc is illuminated, when they are in their superior conjunction; and they are more or less illuminated, as they are nearer or farther from their superior conjunction.

When Venus, or Mercury, is in its superior conjunction, the whole of its enlightened hemisphere is towards the earth, and its entire disc is visible: as it passes towards its inferior conjunction, its enlightened hemisphere turns, by degrees, from the earth, till, at the inferior conjunction, it is wholly turned from the earth, and the planet becomes invisible.

## P R O P. LXV.

If an inferior planet is in one of its nodes at the time of its inferior conjunction, it will appear as a spot in the disc of the sun.

When the planet is in the nodes, it will be in the plane of the ecliptic; and if at the same time it be in its inferior conjunction, it will neither appear above nor below the sun, as it does when in conjunction in other parts of its orbit, but on the sun's disc.



## C H A P. IV.

*Of the SUPERIOR PLANETS, MARS, JUPITER, and SATURN.*

DEF. L. The superior planets are those whose orbits are farther from the sun than the orbit of the earth, namely, Mars, Jupiter, and Saturn.

## P R O P. LXVI.

The superior planets are sometimes in conjunction with the sun, sometimes in opposition, and sometimes in quadrature.

Because the orbit of a superior planet lies without that of the earth, in performing its revolution round the sun, it must sometimes be in such a situation, that the earth shall be between it and the sun, when it will appear on the opposite side of the heavens, or be in opposition: sometimes <sup>the sun</sup> must be between ~~the sun~~ and the earth, when it will be in conjunction, and sometimes in the midway between the opposition and conjunction, or in quadrature.

Let S be the sun; QPO a part of the orbit of Jupiter; P the planet; *adg*, or *nkg*, the earth's orbit. When the earth is at *d*, the sun at S, and the planet at P, the planet is in conjunction: when the earth is at *k*, the sun at S, and the planet at P, the planet is in opposition: when the earth is at *n* or *g*, and the planet at P, the planet will be in quadrature.

Plate 10.  
Fig. 1.

## P R O P. LXVII.

The apparent diameter of a superior planet is greatest when the planet is in opposition.

For, when the planet is in conjunction, its distance from the earth is greater than when it is in opposition, by the diameter of the earth's orbit.

M m

P R O P.



## P R O P. LXVIII.

If a superior planet were at rest, its apparent geocentric motion in any given time would be proportional to the angle subtended by the arc described by the earth in that time, and seen from the planet.

Plate 10.  
Fig. 1.

When the earth is at  $a$ , the planet  $P$  appears in the right line  $aPA$ , and among the stars in the heavens at  $A$ : when the earth is at  $b$ , the planet appears at  $B$ . Therefore while the earth moves from  $a$  to  $b$ , the planet appears to move from  $A$  to  $B$ . But this arc  $AB$  is proportional to the angle  $APB$ , that is, to the opposite angle  $aPb$ , which is the angle which the arc  $ab$  would subtend to an eye placed at the planet  $P$ .

## P R O P. LXIX.

The geocentric velocity of a superior planet is greatest at its conjunction and opposition.

Arcs of a given length near the points  $d$  and  $k$ , when the planet is seen from the earth in conjunction or opposition, would appear greater than arcs in any other part of the earth's orbit, viewed from the planet  $P$ , because the former are seen directly, the latter obliquely: consequently, these arcs would subtend greater angles; whence the apparent velocity of the planet, as viewed from the earth, is greater at the conjunction or opposition, than at any other time.

## P R O P. LXX.

When a line drawn from a superior planet to the earth is a tangent to the earth's orbit, the planet appears stationary.

While the earth is near  $a$  or  $g$ , points in its orbit lying in a line which, drawn from the planet, is a tangent to that orbit, the arc which it then describes so nearly coincides with the tangent line, that the earth for some time seems to move directly from or towards the planet in the line of the tangent; whence the planet must, during that time, appear to be in that line, or to be stationary.



## P R O P. LXXI.

When a superior planet is passing from one of its stationary situations to the other through the conjunction, its geocentric motion is direct; when through the opposition, retrograde.

While the earth is moving from *a* through *d* to *g*, the sun and planet being both on the same side of the earth, the motion of the earth will produce an apparent motion in the sun and planet the same way, and both will appear to move from *A* towards *G*. But while the earth moves from *g* to *n* through *k*, the sun and planet being on contrary sides of the earth, the motion of the earth in its orbit will produce an apparent motion of the sun and planet in contrary directions, that is, whilst the sun appears to move from west to east, the planet will appear to move from east to west in the direction *G*, *H*, *I*, &c.

Plate 10.  
Fig. 1.

SCHOL. The progress, regress, and stations of the superior planets, must take place notwithstanding the motion of the earth, since this only changes the place of the conjunctions and oppositions, on which these depend, as was shewn Prop. LX. Schol.

EXP. Represent the direct and retrograde motions of the superior planets on the orrery.

## P R O P. LXXII.

Mars sometimes appears round, sometimes gibbous; Jupiter and Saturn always appear round.

When Mars is either in opposition or conjunction, his whole illuminated hemisphere is towards the earth, but when it is in quadrature, some part of his illuminated disc is turned from the earth. The same must happen in the revolutions of Jupiter and Saturn about the sun; but their great distance renders the difference between the perfect and imperfect illumination of their disc imperceptible.



## C H A P. V.

## OF THE MOON.

## S E C T. I.

*Of the VARIATIONS in the APPEARANCE of the MOON.*

DEF. LI. When the moon is at its greatest distance from the earth in its orbit, which is elliptical, or at its higher apsis, it is said to be in its *Apogee*; when at its least distance, or lower apsis, in its *Perigee*.

DEF. LII. When the moon is in conjunction with the sun, it is *New Moon*; when in opposition, it is *Full Moon*: its conjunction and opposition are called by the common name of its *Syzygies*.

DEF. LIII. A *Periodical Month* is the time in which the moon describes its orbit; a *Synodical Month* is the time which passes between one new moon and the next.

## P R O P. LXXIII.

A synodical month is longer than a periodical month.

Because the moon moves in the same direction with the sun, while the moon performs one revolution in its orbit, the sun, by its apparent annual motion, is advanced in the ecliptic; consequently, the moon must pass beyond the point in which it has completed its revolution before it comes again into conjunction with the sun.

Plate 10.  
Fig. 8.

Let S be the sun, BA a part of the earth's orbit, *md*, MD, the diameter of the moon's orbit when the earth is at B, or A. While the earth is at A, if the moon be at D, it will be in conjunction, and if the earth continued in the same place, after one revolution in its orbit, it would be again in conjunction: but if, during the revolution of the moon, the



the earth is removed to B, the moon at the end of the revolution will be at  $d$ , a point which is not between the earth and sun; it must therefore move on from  $d$  to  $e$  before it will be in conjunction.

## P R O P. LXXIV.

The moon, at its conjunction, is invisible; at its opposition, its whole disc is enlightened; at its quadratures, it is half enlightened; between the conjunction and quadrature, it is horned; and between the quadrature and opposition, it is gibbous.

Let QTL be a part of the earth's orbit, S the sun, T the earth, ACEG the moon's orbit. When the moon is at E, or in conjunction, because it is between the earth and sun, its illuminated hemisphere will be wholly turned from the earth, consequently, its disc will be dark. At A, being in opposition, its illuminated hemisphere will be wholly towards the earth, and its whole disc will be visible. At C, or G, the apparent distance of the moon from the sun will be 90 degrees, for a right line from C or G, will make a right angle with the line TS, in which the sun appears: whence the moon at each of them is half way between the opposition and conjunction, that is, in the middle state between the perfect illumination and the entire darkness of its disc; consequently, its disc is half enlightened. In passing from C to E, and from E to G, its disc will be more than half illuminated, or it will appear gibbous; in passing from G to A, and from A to C, its disc will be less than half illuminated, or it will appear horned.

Plate 10.  
Fig. 2.

DEF. LIV. A circle supposed to be drawn upon the surface of the moon, separating the illuminated from the dark hemisphere, is called the *Circle of Illumination*: a circle which separates its visible from its invisible hemisphere, is called the *Circle of the Disc*.

## P R O P. LXXV.

If the centers of the sun, the earth, and the moon, are joined by straight lines forming a triangle, the external angle of this triangle at the moon, is equal to the angle contained between the circle of illumination, and the circle of the disc.

Let S be the sun, T the earth, O the <sup>center of the</sup> moon, and  $mrnq$  the moon's orbit. Let STO be the supposed triangle. Draw the line  $rq$  perpendicular to SO, and  $nm$  perpendicular to TOP. The angle SOP will be equal to the angle  $mOq$ , contained between  $rq$ , which represents the circle of illumination, and  $nm$ , which represents the circle of the disc.

Plate 10.  
Fig. 3.

Because



Because the angles  $SO_r$ ,  $TO_m$ , are by construction right angles, they are equal: and  $PO_r$ ,  $qOT$ , are vertical angles, and therefore equal. Consequently, the remaining angles  $POS$ ,  $mOq$ , are equal.

Plate 10.  
Fig. 3. SCHOL. This Proposition serves to explain all the different phases of the moon. For example; when the moon is moved from  $O$  to  $A$ , the line  $SO$  coincides with  $SA$ , and  $TO$  with  $TA$ ; therefore  $TO$ ,  $OS$ , lie in the same line, and the external angle  $POS$  is nothing: whence the two circles of illumination and of the disc coincide; and because the disc is then turned from the sun, it is wholly dark. When the moon is in quadrature, the line  $TOP$  will be a tangent to the moon's orbit; whence  $SOP$  will be a right angle, and the two circles will be at right angles to each other, and the disc will appear half illuminated. If the angle  $POS$  be less or greater than a right angle, the circle of illumination will make an angle with that of the disc, less or greater than a right angle: whence the illuminated part will appear horned or gibbous. Lastly, when the moon is in opposition, the lines  $SO$ ,  $ST$ , become one and the same line; whence the circles coincide, and the whole disc is illuminated.

EXP. Represent the changes of the moon upon the orrery.

#### P R O P. LXXVI.

The horns of the moon, before the conjunction, are turned towards the east; after it, towards the west.

The sun, after the conjunction, setting before, that is, to the west of the moon, illuminates the west side of the moon's disc; whence its horns, which are towards the dark part of the disc, are towards the east. The reverse takes place at the opposition, when the moon is seen in the east, before the sun rises.

#### P R O P. LXXVII.

When the moon is horned, its obscure part is visible by the reflection of the rays of the sun from the earth.

Plate 10.  
Fig. 2. When the moon is at  $D$  or  $F$ , near the conjunction, the enlightened hemisphere of the earth will be towards the moon, and reflect the rays of the sun upon it.

#### P R O P. LXXVIII.

The moon always has nearly the same side towards the earth.

This is proved by observation.

COR. Hence, if the moon revolves about its axis, its periodical time must be equal to that of its revolution in its orbit round the earth.

P R O P.



## P R O P. LXXIX.

The moon appears to have two librations, one upon a line perpendicular to its axis, called its libration in latitude; the other upon its axis, called its libration in longitude.

This appears from observation, some small portions of the surface of the moon being visible in some parts, and invisible in other parts, of its orbit.

## P R O P. LXXX.

The librations of the moon may be explained on the supposition that the moon has a revolution round its axis.

Let IH represent the plane of the moon's orbit, E the earth, and CMD the moon; and let AB be the axis round which the moon revolves, and A be called its north pole, and B its south pole. CMD will in this situation be the visible hemisphere, and CD the plane of the disc. By the libration in latitude the line AB appears to vibrate on the line IH, so that sometimes the point A is visible, and sometimes the point B. This variation attends the moon's revolution in its orbit: in one half of the orbit, the pole A is always visible, and in the other half the pole B. It must therefore arise from the inclination of the axis of the moon to the plane of its orbit. When the moon is at I, if the axis AB is inclined to IH the plane of the orbit, making with it the angle AIH to a spectator at E, the visible hemisphere will be CND, and the pole A will be within the disc: but when the moon is at H, the visible hemisphere will be CMD; and, the axis AB being always parallel to itself, the pole B will be within the disc.

Plate 10.  
Fig. 4.

Again, let the moon be at A, the earth at T, and a line  $dcs$ , passing through the center of the moon, be its axis. If a circle, whose plane is perpendicular to  $Tc$ , pass through the line  $dcs$ , this circle will be that of the disc, and  $dpqs$  will be the visible hemisphere. But the moon has an apparent motion, by which the disc is changed, so that at one time  $rb$  is the circle of the disc, and  $rqb$  the visible hemisphere, at another  $fq$ , and  $fcq$ . In the former case,  $sr$  becomes visible, and  $db$  invisible, in the latter case,  $df$  becomes visible, and  $qs$  invisible. This libration in longitude arises from the elliptical form of the moon's orbit. If the moon has a rotation round its axis, it has been shewn that its revolution will be completed in the time of one revolution in its orbit; and because the motion round the axis is uniform, one quarter of a revolution will be completed in one fourth part of the periodical time. But, the moon's orbit being supposed elliptical, and the earth placed at T one of the foci, the moon will move slower at A its apogee, than at P its perigee, and its velocity will continually increase in moving from A to P, and decrease in moving from P to A. If therefore when the moon is

Fig. 5.

at



Plate 10.  
Fig. 5.

at A,  $dbqs$  is the visible hemisphere, after it has moved from A to B, through that quarter of its orbit, in which it moves with its less velocity, and consequently takes up more than a quarter of its periodical time,  $ds$  will not be perpendicular to  $Tc$ , but the point  $s$  will have turned from west to east more than a quarter round the axis O: hence the point  $s$  will not be visible when the moon is at B, and  $bl$  instead of  $dc$  will be the circle of the disc. In passing from B to P, its excess of velocity will make up for its defect in passing from A to B; and at B it will have completed half its orbit in half its periodical time: but in half its periodical time, it will have revolved half round its axis, therefore at P,  $ds$  will again be perpendicular to  $Tc$ , and  $dxs$  will again be the visible hemisphere. The reverse will take place in passing from P to A through D, when  $lb$  will again be the circle of the disc, and  $s$  will be within it.

#### P R O P. LXXXI.

The moon revolves about its axis in the same time in which it revolves about the earth.

Plate 10.  
Fig. 4.

Fig. 5.

Without such a revolution, the phenomena of its librations could not happen. If the point A were visible in one part of the moon's orbit, it would be always visible, without a rotation about an axis oblique to the plane of the orbit, to produce an apparent motion of the point A, or the libration of latitude. If  $ds$  were perpendicular to  $cT$ , when the moon is at A, it would be so in every other part of the orbit; and therefore  $dxs$  would always be the illuminated hemisphere, if there were not a revolution about the axis to produce, in the manner above explained, the libration of longitude. These librations therefore prove the existence of this revolution: and it has been shewn, that if there be such a revolution, its periodical time is the same with that of the moon in its orbit.

#### P R O P. LXXXII.

The orbit of the moon is an ellipse.

It is only on this supposition that the libration of longitude can be explained: from this phenomenon, therefore, the elliptical form of the orbit may be inferred.

#### P R O P. LXXXIII.

The moon's surface is irregular.

If the surface were every where regular, the limit between the enlightened and dark parts of the disc being the circle of light, that is a perfect great circle of a sphere, would be exactly defined when the moon is horned, half enlightened, or gibbous, contrary to observation.

SECT.



## S E C T. II.

*Of ECLIPSES.*

DEF. LV. An *Eclipse of the Moon* happens when the earth, passing between the sun and moon, casts its shadow on the moon.

## P R O P. LXXXIV.

The moon can only be eclipsed at the full, or in opposition.

For it is only when the moon is in opposition that it can come within the shadow of the earth, which must always be on that side of the earth which is from the sun.

## P R O P. LXXXV.

The moon can only be eclipsed when, at the full, it is in or near one of its nodes.

The earth being in the plane of the ecliptic, the center of its shadow is always in that plane; if therefore the moon be in its nodes, that is, in the plane of the ecliptic, the shadow of the earth will fall upon it: also, since this shadow is of considerable breadth, it is partly above and partly below the plane of the ecliptic; if therefore the moon in opposition be so near one of its nodes, that its latitude is less than half the breadth of the shadow, it will be eclipsed. But, because the plane of the moon's orbit makes an angle of more than 5 degrees with the plane of the ecliptic, it will frequently have too much latitude at its opposition to come within the shadow of the earth.

## P R O P. LXXXVI.

The sun is larger than the earth, and the shadow of the earth is a cone, the base of which is on the surface of the earth.

If the earth were larger than, or equal to, the sun, it is manifest, that its shadow would either perpetually enlarge, or be always of the same dimension; but in this case, the

N n

superior



superior planets would sometimes come within it, and be eclipsed, which never happens. Therefore the sun is larger than the earth, and produces a shadow from the earth of a conical form, which does not extend to the orbit of Mars.

## P R O P. LXXXVII.

The moon is eclipsed by a section of the earth's shadow.

Plate 10.  
Fig. 7.

Let CDE be the earth, CME the cone of its shadow, and FH a part of the moon's orbit passing through the shadow; it is manifest that as the moon describes this part of its orbit, it passes through the circular section FGHL.

DEF. LVI. The moon's *Horizontal Parallax*, is the angle which a semidiameter of the earth would subtend, if it were viewed directly from the moon.

## L E M M A.

*Half the angle of the cone of the earth's shadow may be taken as equal to the angle of the sun's apparent semidiameter.*

Plate 10.  
Fig. 6.

Let AFBG be the sun, HED the earth, HMD or BMA the angle of the cone of the earth's shadow, and CMD half this angle. SA, a semidiameter of the sun, drawn from its center to the point of contact of the tangent AM, is perpendicular (El. III. 16.) to AM, and is therefore seen directly from D; and it subtends the angle SDA; it must therefore appear large in proportion to the magnitude of this angle. But in the triangle SDM, the external angle SDA is equal to the two angles CSD, CMD; of which, CSD, the angle in which the semidiameter of the earth is viewed from the sun, is so small that without any sensible error it may be reckoned as nothing, and SDA be said to be equal to SMD.

## P R O P. LXXXVIII.

The semidiameter of the section of the earth's shadow which eclipses the moon, is equal to the difference between the horizontal parallax of the moon, and the sun's apparent semidiameter.

Plate 10.  
Fig. 7.

CT, being a semidiameter of the earth drawn from the point of contact of the tangent CM, is perpendicular to CM. CT will therefore be seen directly, from the point F in the moon's orbit, subtending the angle CFT, which is (Def. LVI.) the moon's horizontal



horizontal parallax. FMG is the semiangle of the cone of the earth's shadow, equal to the angle of the sun's apparent semidiameter, because MC produced would be a tangent to the circle of the sun's disc. FG is the semidiameter of the section FGHL of the earth's shadow through which the moon passes in an eclipse; and FTG the angle which this semidiameter will subtend when it falls upon the moon, and is viewed from the earth. Now the angle CFT is equal to the two angles FMG, FTG (El. I. 32.) consequently, FTG is equal to CFT—FMG; but by the preceding Lemma, FMT may be taken for the angle of the sun's apparent semidiameter; and CFT is the moon's horizontal parallax: whence the Proposition is manifest.

COR. Hence the section of the earth's shadow, by which the moon is eclipsed, is broader than the moon; for the semidiameter of the shadow, by this Prop. is  $61' - 16' = 45'$ . and the diameter of the moon is about  $31'$ .

DEF. LVII. An eclipse of the moon is *partial*, when only a part of its disc is within the shadow of the earth; it is *total*, when all its disc is within the shadow; and it is *central*, when the center of the earth's shadow falls upon the center of the moon's disc.

Plate 10.  
Fig. 10.

#### P R O P. LXXXIX.

The moon, at the full, will not be eclipsed, if its latitude is equal to, or greater than, the sum of its own semidiameter, added to the semidiameter of the earth's shadow.

Because a circle or ellipse appears as a right line when the eye is in the same plane with it, let OO represent the plane of the ecliptic, RR the plane of the moon's orbit, and N the node. At the full moon, if the earth's shadow be at A, the moon F must be in the same part of the heavens, because it is in opposition. But because only one half of the shadow of the earth, or about  $45'$ . is on the same side of the ecliptic with the moon, and only one half of the moon's breadth, or about  $16'$ . is on the side of its orbit towards the earth's shadow; if the center of the moon ● is more than  $61'$ . from the center of the shadow, the moon will pass clear of the shadow, and will not be eclipsed.

Plate 10.  
Fig. 10.

Let GE be an arc of the moon's orbit, AB an arc of the ecliptic, and Cc an arc in a secondary of the ecliptic equal to the moon's latitude. If this arc be equal to, or greater than, Ct and cl, the sum of the semidiameters of the earth's shadow, and of the moon, it is manifest, that the shadow cannot pass over any part of the moon's disc.

Fig. 9.



## P R O P. XC.

The moon, at the full, will be partially eclipsed, if its latitude is less than the sum, but greater than the difference, of its own semidiameter and that of the earth's shadow.

Plate 10.  
Fig. 9.

If  $Cc$  the latitude of the moon be supposed less than  $Ct$ ,  $cl$ , the sum of the semidiameters of the shadow and the moon,  $l$  the lower edge of the moon will be below  $t$  the upper edge of the shadow; whence the side of the moon towards the ecliptic will be eclipsed. But, because  $Cc$  the moon's latitude, added to  $co$  its semidiameter, is greater than  $Ct$  the semidiameter of the earth's shadow, the upper edge of the moon  $o$ , cannot come within the shadow; whence the eclipse will be partial. And because in this case the moon's latitude, together with its semidiameter, is greater than the semidiameter of the shadow, the moon's latitude is greater than the difference of the semidiameters of the shadow and the moon: or, because  $Cc+co$  is greater than  $Ct$ ,  $Cc$  is greater than  $Ct-co$ .

## P R O P. XCI.

The moon will, at the full, be totally eclipsed, if its latitude is less than the difference between its own semidiameter, and that of the earth's shadow.

Fig. 9.

If  $Cc+co$  be less than  $Ct$ , that is, if  $Cc$  be less than  $Ct-cl$ , the upper edge of the moon may come within the shadow of the earth.

## P R O P. XCII.

The moon is centrally eclipsed, only when, in opposition, it is in one of its nodes.

The node  $N$ , being the common intersection of the moon's orbit, and the plane of the ecliptic, is in both. Therefore when the moon is in the node, its center is in the plane of the ecliptic, in which is the center of the earth's shadow, and consequently at the full these centers coincide.

SCHOL. I. Both the moon and the shadow moving from east to west, the moon would always be in eclipse while it was at, or near, its nodes, if it moved with the same velocity as the earth: but because it moves much faster than the shadow of the earth, it soon passes from its opposition out of the shadow.

SCHOL.



SCHOL. 2. The femiangle of the cone of the earth's shadow being known, the length of the shadow may be found.

The femiangle CMT being equal to the sun's apparent femidiameter, or about 16'. and CT a femidiameter of the earth being perpendicular to the tangent CM, if TM be radius, CT is the sine of the angle CMT. Therefore as the sine of an angle of 16'. is to radius, so is CT to TM, the height of the cone; that is, as 1 to 217. Whence the length of the shadow is about 217 femidiameters of the earth.

Plate 10.  
Fig. 7.

P R O P. XCIII.

The moon in a total eclipse, is not wholly invisible.

This is known by observation; and the phenomenon is produced by the reflection of rays of light falling upon the earth's atmosphere, towards the shadow, and consequently towards the moon in the shadow.

DEF. LVIII. An *Eclipse of the Sun* happens when the moon, passing between the sun and the earth, intercepts the sun's light.

P R O P. XCIV.

The sun can only be eclipsed at the new moon.

For it is only when the moon is in conjunction, that it can pass directly between the sun and the earth.

P R O P. XCV.

The sun can only be eclipsed, when the moon, at its conjunction, is in or near one of its nodes.

For unless the moon is in or near one of its nodes, it cannot appear in or near the same plane with the sun; without which it cannot appear to us to pass over the disc of the sun. At every other part of its orbit, it will have so much northern or southern latitude, as to appear either above or below the sun. If the moon be *in* one of its nodes, having no latitude, it will cover the whole disc of the sun, and produce a *total eclipse*, except when its apparent diameter is less than that of the sun: if it be *near* one of its nodes, having a small degree of latitude, it will only pass over a part of the sun's disc, or the eclipse will be partial.

P R O P.



## P R O P. XCVI.

In a total eclipse of the sun, the shadow of the moon falls upon that part of the earth where the eclipse is seen.

Plate 10.  
Fig. 11.

Let SL be the sun, TR the moon, VM a part of the surface of the earth. A spectator placed any where between V and M, will not see any part of the sun, because the moon will intercept all the rays of light which come to him directly from the sun; and it is manifest, that, in this situation, the moon, being an opaque body, will cast its shadow upon VM, the part of the earth where the eclipse is total.

## P R O P. XCVII.

The shadow of the moon is a cone, terminated in a point.

Because (by Prop. LXXXVIII. Cor.) the diameter of the moon is less than the diameter of the earth's shadow, where it eclipses the moon, and this diameter (by Prop. LXXXVI.) is less than the diameter of the earth, the diameter of the moon is much less than that of the sun: consequently, its shadow will be a cone, terminated in a point. The tangents LAR, SAT, terminate in A; and only the rays that would pass within these tangents are intercepted by the moon; therefore RTA is the form of the moon's shadow.

## L E M M A.

*Half the angle of the cone of the moon's shadow is equal to the angle of the apparent semidiameter of the sun.*

Plate 10.  
Fig. 6.

Let FBGA be the sun; HED the moon; the cone HMD the moon's shadow; CMD the semiangle of this cone; SA the semidiameter of the sun, and SDA the angle which this semidiameter would subtend, if viewed from D. It may be proved, as in Prop. LXXXVIII. *Lemma*, that CMD is equal to SDA; for the distance of the moon from the sun is so nearly equal to that of the earth from the sun, that the apparent semidiameter of the sun, as seen from the earth or moon, may be considered as equal.

## P R O P. XCVIII.

A semidiameter of the moon's shadow, where it falls upon the earth, is equal to the difference between the apparent semidiameters of the moon and sun.

Let



Let CDE represent the moon, CME the cone of its shadow; FG the semidiameter of the moon's shadow where it falls upon the earth in a solar eclipse; CT the semidiameter of the moon, CFT its angle viewed from F, and FTG the angle of the apparent semidiameter of the moon's shadow viewed from T.

Plate 10.  
Fig. 7.

In the triangle TFM, the external angle TFC (El. I. 32.) is equal to the two angles FTG, FMG. Therefore FTG is equal to the difference between TFC and FMG: and FMG is (by the preceding Lemma) equal to the angle of the sun's apparent semidiameter, and TFC is the angle of the moon's apparent semidiameter: whence the Proposition is manifest.

### P R O P. XCIX.

In a partial eclipse of the sun, a *penumbra*, or imperfect shadow of the moon, falls upon that part of the earth where the partial eclipse is seen.

A spectator at N, or P, might see the whole sun; for a ray passing from the most remote side of the sun, S or L, would not be intercepted by the moon. But at any point in NM, VP, the spaces between the moon's shadow and the points N, P, the spectator would only see a part of the sun: thus at G, or D, he would see that half of the sun which lies without the tangents DRC, GTC: consequently, in all places between the points N, M, and P, V, there will be less light from the sun, than if it were not at all eclipsed. This deficiency of light is called the moon's *penumbra*.

Plate 10.  
Fig. 11.

### P R O P. C.

The moon's penumbra is an increasing cone, and its darkness increases towards the shadow of the moon.

While a spectator is in the space between N and M, or P and V, he is in the penumbra; but at the points N, P, passes out of it. Therefore the tangents NS, PL, are the limits of the penumbra. If tangents be supposed drawn round the spherical surfaces of the sun and moon, they will form two cones, having their common vertex at F, and increasing, the one towards the points L, S, the other towards the points N, P. And as the spectator moves from N towards M, or from P towards V, a greater and greater portion of the sun continually becomes invisible to him; whence the penumbra increases in darkness towards M and V.

Plate 10.  
Fig. 11.

P R O P.



## P R O P. CI.

The femiangle of the moon's penumbra is equal to the angle of the sun's apparent semidiameter.

Plate 10.  
Fig. 11.

Let  $SL$  be the sun, and  $TR$  the moon. By Prop. C. the tangents  $SFN$ ,  $LFP$ , terminate the cone of the penumbra.  $CE$ , drawn from the center of the sun to that of the moon, bisects  $TFR$ , the angle of this cone; whence  $EFT$  is its femiangle.  $LC$  being the semidiameter of the sun,  $LTC$  is the angle under which this semidiameter would appear from the moon  $T$ . I say,  $TFE$  is equal to  $LTC$ . For, in the triangle  $TCF$ , the external angle  $EFT$  is equal to the two internal and opposite angles  $FTC$ ,  $FCT$ , that is, to the two angles  $LTC$ ,  $TCE$ . Therefore  $EFT$  is equal to  $LTC$ ,  $TCE$ . But  $TCE$ , being the angle which the moon's semidiameter would subtend, if viewed from the sun, is so small that it may be neglected; ~~and~~ whence  $EFT$  may be considered as equal to  $LTC$ .

## P R O P. CII.

The semidiameter of the moon's penumbra, in that part through which the earth passes in an eclipse of the sun, is equal to the sum of the apparent semidiameters of the sun and moon.

Plate 10.  
Fig. 12.

Let  $CD$  be the moon,  $CDAB$  its penumbra, and  $CMD$  the angle of the cone of the penumbra: and let  $AEBF$  be the section of the penumbra through which the earth passes in an eclipse of the sun,  $AB$  its diameter, and  $AT$  its semidiameter.  $MLT$  drawn from the vertex through the center of the moon, will bisect the angle  $CMD$ . Therefore (by Prop. CI.)  $CML$  the femiangle of the penumbra is equal to the sun's apparent semidiameter. And  $CL$  is the moon's apparent semidiameter, as seen from the earth  $A$ , subtending the angle  $CAL$ ; and  $AT$  the semidiameter of the penumbra, seen from  $L$ , subtends the angle  $ALT$ . Now the angle  $ALT$  is equal to the two angles  $CML$ ,  $CAL$ ; whence the truth of the Proposition is manifest.

DEF. LIX. The disc of the earth, is that hemisphere of the earth, which is seen, as a circle, from the moon.

## P R O P. CIII.

At the new moon the whole disc of the earth is enlightened.

For, the moon being then between the sun and the earth, the earth, viewed from the moon, will appear in opposition, and consequently its enlightened hemisphere will be towards the moon.

P R O P.



## P R O P. CIV.

The semidiameter of the earth's disc, is equal to the moon's horizontal parallax.

The moon's horizontal parallax is the apparent semidiameter of the earth, as viewed from the moon, that is, it is equal to the semidiameter of the disc, since the disc is a hemisphere of the earth viewed from the moon.

## P R O P. CV.

If the latitude of the moon, when new, is equal to, or greater than, the sum of the semidiameter of the penumbra and the moon's horizontal parallax, there will be no eclipse of the sun; if less, there will be an eclipse, either partial, or total.

Let ACB be a part of the ecliptic; let GNE be the plane of the moon's orbit; let N be the node, DL the earth's disc, and  $ol$  a section of the moon's penumbra. Because the center of the moon's penumbra is always in a right line passing through the centers of the sun and the moon, the distance of  $c$ , the center of the moon's penumbra, from the plane of the ecliptic must always be the same with the distance of the center of the moon from the same plane, that is, with the latitude of the moon. And because the center of the earth, or the earth's disc C, is in the ecliptic, Cc the distance between the centers of the penumbra and of the earth's disc, is always equal to the latitude of the moon. Now, if Cc, or the latitude of the moon, be equal to, or greater than,  $cl$  the semidiameter of the penumbra, together with  $tC$  the semidiameter of the earth's disc, or the moon's horizontal parallax, then no part of the penumbra will fall upon the disc, that is, there will be no eclipse. If Cc the latitude of the moon be less than  $tC + l$  the edge of the penumbra will be nearer the ecliptic than the edge of the disc, and there will be a partial eclipse. And if Cc be less than C $t$ , the shadow of the moon will pass over some part of the disc of the earth, and where this happens, the eclipse will be total.

Plate 10.  
Fig. 9.

## P R O P. CVI.

If the moon, when new, is in one of its nodes, the eclipse of the sun will be central.

For then the centers of the earth, sun and moon being all in the plane of the ecliptic, the center of the moon will pass between the sun's center, and that of the earth.

O o

SCHOL.



SCHOL. 1. The penumbra of the moon in a central eclipse, will not cover the whole disc of the earth.

The femidiameter of the moon's penumbra, being equal to the sum of the apparent femidiameters of the sun and moon, that is, about  $16'. 23''.$  +  $15'. 37''.$  or  $32'.$  at the medium; its diameter is about  $64'.$  whereas the diameter of the earth's disc is about  $120'.$  whence the penumbra cannot cover the whole disc.

SCHOL. 2. The height of the shadow of the moon is about  $60\frac{1}{2}$  femidiameters of the earth. The semiangles of the earth's shadow, and of the moon's shadow, being each equal to the sun's apparent <sup>semidiameter</sup> diameter, the angles are equal to one another, and these cones are similar. Therefore as the femidiameter of the base of the earth's shadow (that is, of the earth) is to the femidiameter of the base of the moon's shadow (that is, of the moon), so is the height of the earth's shadow to the height of the moon's shadow. Now the femidiameter of the earth is to that of the moon nearly as 100 to 28, and the height of the earth's shadow is about 217 femidiameters of the earth: whence the height of the moon's shadow is equal to about  $60\frac{1}{2}$  femidiameters of the earth; for  $100 : 28 :: 217 : 60\frac{1}{2}$  nearly.

DEF. LX. An eclipse of the sun is said to be *annular*, when at the time of the eclipse, a ring of the sun appears round the edges of the moon.

#### P R O P. CVII.

A central eclipse of the sun will be an annular one, if the distance of the moon from the earth at the time of the eclipse be greater than its mean distance.

Plate 10.  
Fig. 11.

SL being the sun, TR the moon, TAR the moon's shadow, and EA the height of this shadow, which is about  $60\frac{1}{2}$  femidiameters of the earth; if at the time of a central eclipse PN is a part of the surface of the earth, those who live in the parts PV, MN, being in the penumbra, will (by Prop. XCIX.) see a partial eclipse; and those who live between V and M, being in the shadow, will (by Prop. XCVI.) see a total eclipse. But if the distance of the moon from the earth be equal to EA, or  $60\frac{1}{2}$  femidiameters of the earth (which is the moon's mean distance) A will be the only point from which the eclipse will appear total. And if the moon's distance be greater than EA, as EO, the shadow not reaching the earth, there will be no total eclipse. Consequently, though a spectator at O would see a central eclipse (because the centers C, E, are in the same line with the point of vision S), yet the eclipse would not be total, because the spectator is not in the shadow of the moon. Hence it must appear annular: for let ORX be a tangent to the moon drawn from the eye at O; it will fall upon the sun at X, and the  
part



part XL of the sun will be visible: in like manner, parts of the sun equal to XL will be visible all round the moon, forming a ring.

COR. Hence it appears, that in an annular eclipse it is the penumbra of the moon which falls upon the earth.

DEF. LXI. The *Lunar Ecliptic Limit*, is the least distance that the moon can be at from one of its nodes, without being eclipsed at the time of opposition: the *Solar Ecliptic Limit*, is the least distance the moon can be at from one of its nodes, without eclipsing the sun at the time of conjunction.

### P R O P. CVIII.

The solar ecliptic limit is greater than the lunar.

By Prop. CV. it is found, that  $92'$  is the least latitude the moon can have, when new, without eclipsing the sun. If therefore  $Nc$  be the distance of the moon from the node, when its latitude, or  $cC$ , is  $92'$ . in the triangle  $CcN$ , the angle  $cCN$  being a right angle, because  $cC$  is perpendicular to the plane of the ecliptic, the angle  $cNC$  being about  $5^\circ. 30'$ . the inclination of the moon's orbit to the plane of the ecliptic; and the side  $cC$  being  $92'$ .  $Nc$  will be found by trigonometry to be about  $16^\circ$ . In the same manner, supposing  $Cc$  the latitude of the moon to be  $61'$ . according to Prop. LXXXIX. the length of the side  $Nc$ , or the lunar ecliptic limit, will be found to be about  $12^\circ$ . Whence the truth of the Proposition is manifest.

Plate 10.  
Fig. 9.

COR. There are more eclipses of the sun, in a course of years, than of the moon; for, the sun will be eclipsed, if, when the moon is new, it is within  $16^\circ$ . of one of the nodes, but the moon only when at the full it is within  $12^\circ$ . of one of the nodes; the sun may be eclipsed while the moon is in 64 degrees of its orbit, but the moon only while it is in 48 degrees of its orbit.

SCHOL. Every eclipse of the moon will be visible, if the moon be above the horizon at the time of the eclipse; because that part of the moon on which the shadow of the earth falls, must appear obscured wherever the disc of the moon is visible. But the sun may be eclipsed, and yet the eclipse be invisible in places to which the sun is above the horizon, because there can be no eclipse of the sun, except in those parts of the earth which are within the shadow or penumbra of the moon, and neither of these are large enough to cover the whole disc. Hence, in any given place, more eclipses of the moon than of the sun will be seen in a course of years: for though there are more eclipses of the sun than



of the moon, many of the former are not visible at any one place while the sun is above the horizon; but all the latter are visible at the same place while the moon is above the horizon.

## P R O P. CIX.

When the moon is near the first of Aries, and is moving towards the tropic of Cancer, the time of its rising will vary but little for several days together.

If the moon were to move in the equator, its motion in its orbit, by which it describes a revolution, in respect of the sun, in 29 days 12 hours, would carry it every day eastward from the sun about  $12^{\circ}. 11'$ . whence its time of rising would vary daily about 50 minutes. But, because the moon's orbit is oblique to the equator, nearly coinciding with the ecliptic, different parts of it make different angles with the horizon, as they rise or set, those parts which rise with the smallest angles setting with the greatest, and the reverse. Now the less this angle is, the greater portion of the orbit rises, in the same time. Consequently, when the moon is in those parts which rise or set with the smallest angles, it rises or sets with the least difference of time, and the reverse. But in northern latitudes, the smallest angle of the ecliptic and horizon is made when Aries rises and Libra sets, and the greatest when Libra rises and Aries sets; and therefore, when the moon rises in Aries, it rises with the least difference of time. Now the moon is in conjunction in or near Aries, when the sun is in or near Libra, that is, in the autumnal months; when, the moon rising in Aries, whilst the sun is setting in Libra, the time of its rising is observed to vary only two hours in 6 days in the latitude of London. This is called the harvest-moon.

Exp. Let small patches be placed on the ecliptic of a globe, as far from one another as the moon moves from any point of the ecliptic in 24 hours, that is, about  $13\frac{1}{2}$  degrees; then, while the globe is turned round, observe the rising and setting of the patches in the horizon; the hour index will shew the difference of time at which the moon rises or sets in different parts of its orbit.



## C H A P. VI.

*Of the SATELLITES of JUPITER and SATURN.*

## P R O P. CX.

Any fatellite is at its greatest elongation from its primary, when a line drawn from the earth through the fatellite, is a tangent to the orbit of the fatellite.

Let FIE be a part of the orbit of the primary planet, AXBT the earth's orbit, S the sun, KGNL the orbit of a fatellite. If the earth is at X, and the fatellite at L or N, so that a line XL or XN drawn from the earth, is a tangent to the orbit KGNL, it may be shewn, as before concerning the planets, that L or N is the greatest elongation of the fatellite.

Plate 10.  
Fig. 14.

## P R O P. CXI.

Any fatellite appears in inferior conjunction with its primary, when the fatellite is between the earth and the primary, and in superior conjunction, when the primary is between the fatellite and the earth.

If the earth is at X, and the planet at I, the outermost fatellite will be in conjunction with its primary when they both appear in the same line MIV, and in its inferior conjunction at M, and its superior at V.

## P R O P. CXII.

The apparent motion of any fatellite, as it passes from its greatest elongation on one side of its primary through the superior conjunction, to its greatest elongation on the other side, is direct.

As the fatellite passes from N, its greatest elongation on one side, through V its superior conjunction to L, its geocentric motion is from west to east, or *in consequentia*, as was shewn concerning the planets.

P R O P.



## P R O P. CXIII.

The apparent motion of any fatellite, as it passes from its greatest elongation on one side of its primary through the inferior conjunction, to its greatest elongation on the other side, is retrograde.

As the fatellite passes from L through M to N, its geocentric motion will be from east to west, or *in antecedentia*, as was proved concerning the planets.

COR. The fatellites are sometimes to the west and sometimes to the east of their primaries.

## P R O P. CXIV.

The greatest elongations of a fatellite on each side are equal.

For by observation it is found, that the angles LXI, NXI, are equal with respect to all the fatellites.

COR. Hence it appears that the orbits of the fatellites are circular, or nearly so, having their primaries at the center of their orbits.

## P R O P. CXV.

The fatellites of Jupiter and Saturn are eclipsed by their respective primaries.

Plate 10.  
Fig. 14.

The planet I being an opaque body, casts a shadow IV, opposite to the sun. Therefore, when one of its fatellites in describing the arc HOP comes to H, it will be eclipsed by falling into this shadow. If the earth is at B in its orbit, a spectator from the earth will lose sight of the fatellite, when it is thus eclipsed at V; and then as it emerges from the shadow it becomes again visible, till, at P, it passes behind its primary. If the earth be at X, the fatellite will be eclipsed and in occultation at the same time.

## P R O P. CXVI.

When one of the fatellites passes between the sun and its primary, it eclipses the sun.

A fatellite at M will be between the sun and its primary, and occasion an eclipse of the sun on that part of the primary where the shadow of the fatellite passes, which shadow will appear as a dark spot on the disc of the planet to an inhabitant of the earth.

P R O P.



## P R O P. CXVII.

A ray of light is about 8 minutes in coming from the sun to the earth.

Let A be the sun, BECD the earth's orbit, F the planet Jupiter, and HNG the orbit of its inner satellite. Let FGH represent the shadow of Jupiter. While the satellite is between H and G, it is eclipsed; when it comes to H, it emerges, and becomes visible to a spectator at B. From comparing the times of the apparent entrance and emergence of the satellite, with tables calculated for the mean distances of the earth from the satellite, the visible emergence at the least distance is found to happen about 8 minutes sooner, and at the greatest distance about 8 minutes later than by the tables: consequently, the ray of light is about 16 minutes in passing through the earth's orbit, or 8 minutes in coming from the sun to the earth.

Plate 10.  
Fig. 13.

## P R O P. CXVIII.

Jupiter is surrounded by cloudy substances, subject to frequent changes in their situation and appearance, called his Belts. Saturn is encompassed with a Ring.

These are known from observation. The Belts of Jupiter are sometimes of a regular form; sometimes interrupted and broken; and sometimes not at all to be seen. The plane of Saturn's ring is inclined to the plane of the ecliptic at an angle of 31 degrees; which appears like two arms to the planet, and which is only visible when the sun and the earth are both on the same side of its plane. On account of its inclination, it always appears oblique to the eye, and therefore elliptical: whence the part behind Saturn is invisible, and the part before cannot be distinguished from the planet. The ring being opaque, can only be visible when the sun's rays are reflected from its broad surface to the earth, that is, when the sun and the earth are both on the same side of the plane of the ring.

Plate 10.  
Fig. 15.



## C H A P. VII.

## O F C O M E T S.

## P R O P. CXIX.

Comets are opaque and solid bodies.

A comet, at a given distance from the earth, shines much brighter when it is on the same side of the earth with the sun, than when it is on the contrary side; from whence it appears that it owes its brightness to the sun. Comets have been observed to approach so near the sun, that their great heat (computed to be to that of red hot iron as 2000 to 1) must have entirely dissipated them, if they had been other than fixed and solid bodies.

## P R O P. CXX.

The comets describe very eccentric ellipses about the sun, placed in one of their *foci*.

They are observed to approach towards, and afterwards recede from, the sun, and to describe paths in the heavens, which agree with elliptic orbits: it is therefore most probable, that, agreeably to the general analogy of nature, they move in such orbits, and have the sun in one of the *foci* of the ellipse. The calculations framed upon this supposition, by which the returns of comets have been foretold, having, as far as observations have been made, been found to agree with the phænomena, strongly confirm the truth of the Proposition.

SCHOL. I. Comets are often accompanied with a luminous train, called the tail, which is conjectured to be smoke rising from the body in a line opposite to the sun. The body of the comet is supposed to be surrounded by an atmosphere; the sun is also supposed to be surrounded by an ether, or a subtile fluid, extending to a great distance from the sun, which may be considered as the solar atmosphere. From the heat which the comet has acquired by approaching towards the sun, and by the reflection of the sun's rays from the solid body and atmosphere of the comet, the parts of the solar atmosphere where the comet passes are more heated, and consequently more rarefied or specifically lighter than elsewhere. The parts thus rarefied will be put into motion; and since there will be  
a constant



a constant succession of fresh portions of the sun's atmosphere within that of the comet, there will be a perpetual stream of this rarefied matter. This stream will impel the particles of the comet's atmosphere, and make them move along with it, thus producing the smoke which, reflecting the sun's rays, forms the visible tail. And this stream of rarefied solar atmosphere will move those parts of this atmosphere which have the least specific gravity, that is, directly from the sun.

SCHOL. 2. Of all the comets, the periods of only three are known with any degree of certainty. The first of these comets appeared in the years 1531, 1607, and 1682; and is expected to appear every 75th year. The second of them appeared in 1532 and 1661, and may be expected to return in 1789, and every 129th year afterwards. The third, having last appeared in 1680, and its period being no less than 575 years, cannot return until the year 2225. This comet, at its greatest distance, is about 11 thousand two hundred millions of miles from the sun; and at its least distance from the sun's center, which is 49,000 miles, is within less than a third part of the sun's semidiameter from his surface. In that part of its orbit which is nearest the sun, it moves at the rate of 880,000 miles in an hour.



## C H A P. VIII.

## OF THE SUN.

## P R O P. CXXI.

The spots which appear upon the sun's disc, adhere to its surface.

Plate 10.  
Fig. 16.

If one of these spots appears upon the eastern limb or edge of the sun's disc, it moves from thence towards the western edge, and arrives at the western edge in about  $13\frac{1}{2}$  days. Here the spot disappears; and in about  $13\frac{1}{2}$  days more, it is seen again upon the eastern edge; and so continues to go round completing its apparent revolution in 27 days; during one half of which time we see it on the disc of the sun, and during the other half it disappears; which could not happen, if the spots did not adhere to the surface of the sun. Let A be the center of the sun's disc, C its eastern, and D its western edge; HEG the orbit of an opaque body moving round it, and B the eye of the spectator at the earth. If two lines BD and BC are supposed to be drawn from the spectator's eye B, so as to touch the sun at D and C, then DBC, the angle contained between these lines, is the angle under which the sun's diameter appears to a spectator on the earth. EG is the only part of the supposed body's orbit that is within this angle DBC, and consequently, if the body was in any other part of its orbit, except EG, it would not appear upon the sun's disc. But EG is less than half its orbit; and the body would not take up half the time of a revolution to describe EG. Therefore such a body would not be seen upon the sun's disc, as the spots are for half the time of a revolution. But if the orbit HEG is not greater than LDFC, or is close to the sun, that is, if the spot adheres to the sun's surface, then half its orbit DFC, will be within the angle DBC, and consequently the spot will appear upon the sun's disc during one half its revolution; but during the other half of its revolution, whilst it describes CLD, it will disappear, because then it will be behind the sun, and so will be concealed from the earth, which agrees with the phenomena.

## P R O P. CXXII.

The sun is a spherical body, which revolves upon its axis from west to east.

The spots which appear in the sun's disc, adhere to its surface (by Prop. CXXI.) and these spots revolve; therefore the sun revolves round its axis.

Whatever



Whatever side of the sun is turned towards the earth in this rotation, it always appears to be a flat bright circle: but all the sides of it could not appear in this manner unless it was a sphere: therefore the sun is a spherical body.

SCHOL. A real revolution of a spot, and consequently of the sun round its axis, is completed in 25 days, two days less than its apparent revolution, in consequence of the earth's motion in its orbit in the same direction in which the spot moves.

## P R O P. CXXIII.

The axis of the sun is inclined to the plane of the ecliptic.

Each spot upon the sun must describe a circle round the sun, either coinciding with its equator, or parallel to it. If therefore the sun's axis were perpendicular to the plane of the ecliptic, the plane of the sun's equator would be in the plane of the ecliptic; and a spectator on the earth, whose eye is in the ecliptic, would see the spots describing right lines, either in the sun's equator, or parallel to it: but the spots are sometimes seen to describe lines oblique to the plane of the ecliptic: therefore the axis of the sun is inclined to the plane of the ecliptic. This inclination is observed to be an angle of about  $82\frac{1}{2}$  degrees. When the sun's equator crosses the plane of the ecliptic, the spots appear to describe right lines parallel to the sun's equator.



## C H A P. IX.

*Of the* PARALLAXES, DISTANCES, *and* MAGNITUDES *of the*  
HEAVENLY BODIES.

DEF. LXII. The *Parallax* of the heavenly bodies, is the change of their apparent situation with respect to each other, as the spectator views them from different stations.

DEF. LXIII. The *Diurnal Parallax*, is the distance between the apparent place of a heavenly body, as viewed from the surface of the earth, and its apparent place, as viewed from the center of the earth.

Plate 10.  
Fig. 17.

Let DAB be the earth, C its center, A the station of a spectator on the surface of the earth; and F, G, H, different places of the moon, or any other heavenly body: TO, NM, LI, are its different parallaxes, and THO, or AHC; MGN, or AGC, &c. angles of parallax.

SCHOL. If a spectator in his first station at A, sees a planet at G, its apparent place in the heavens will be N; if now, by the diurnal rotation of the earth, he comes into the station P, the planet will appear at M, which is the place in which it would have appeared if viewed from C the center: thus, in all cases, the parallax which arises from the diurnal motion is the same which would arise from a change of station from the surface to the center; for in either case, the change of the spectator's line of view is the same. Hence appears the propriety of the above definition of the *diurnal parallax*.

## P R O P. CXXIV.

The parallax of any planet is always proportional to the angle which a semidiameter of the earth, drawn from the station of the spectator upon the surface to the center, would subtend, if viewed from the planet.

Plate 10.  
Fig. 17.

If the planet be at H, and the spectator at A, AHT will be his line of view; on changing the station of the spectator from A to C, the line of view will become CHO: whence  
TO



TO will be the parallax. But TO subtends and is proportional to THO, or (El. I. 15.) AHC, the angle which the earth's semidiameter would subtend, if viewed from the planet H.

## P R O P. CXXV.

The parallax of a planet depresses its apparent place, by the parallactic arc.

If the planet be viewed from C, its apparent place is O; if from A, its apparent place is T, farther from Z the vertex than O, by the parallactic arc TO.

Plate 10.  
Fig. 17.

## P R O P. CXXVI.

The diurnal parallax of any planet, at a given distance from the earth, is greatest when the planet is in the horizon, and decreases as the altitude of the planet increases.

The parallax (by Prop. CXXIV.) is proportional to the angle which AC would subtend, if seen from the planet H: but this given line, viewed from the given distance of the planet, would continually diminish in its apparent magnitude (by Book IV. Prop. LXXIII.) as the degree of obliquity at which it is viewed increases, that is, as the planet advances from H towards E; therefore the parallax is greatest in the horizon, and decreases as the planet approaches the vertex. The parallactic angle AGC is less than AHC, and AFC less than AGC.

## P R O P. CXXVII.

To find the parallax of the moon, or any planet.

Let HMO be an arc of the horizon; APVM an arc in the meridian; P the elevated pole; V the vertex; E the apparent place of the planet, as seen from the surface of the earth, and S its place, as seen from the center: then ES is the diurnal parallax in the vertical circle VE. Before the planet comes to the meridian, observe its altitude, at E, above the horizon, whence the complement of its altitude, VE, will be known: at the same time observe its distance from the meridian, or its azimuth, EVM. After the planet has passed the meridian, observe when it has the same altitude as at the first observation, that is, when  $\angle V$  is equal to EV. Now, if E is the apparent place of the planet when at the time of the first observation it is viewed from the earth's surface, and S would be its place, at that time, if viewed from the center; and if  $e$  is its apparent place when viewed,

Plate 10.  
Fig. 18.



viewed, at the second observation, from the surface, and  $s$  would be its place, at that time, if viewed from the center; the parallax  $ES$  is equal to the parallax  $es$ , since the altitude was the same at both observations, and consequently  $SV$  is equal to  $sV$ . So that if  $PS$  is the secondary of the equator which passed through the planet at the first observation, and  $Ps$  the secondary which passed through it at the second observation, the planet, between the times of the first and second observation, must have described the arc  $Ss$  in a circle of daily motion. From the time which has passed between the two observations, the arc  $Ss$  (by Prop. XXVII.) may be found, and consequently the angle  $SPs$ . Now, because the angle  $EVM$  is known,  $PVS$  its complement to two right angles is known: and, because at the two observations the planet was at equal altitudes, that is, at equal distances from the meridian, the meridian bisects the angle  $SPs$ , which is known; whence its half  $VPS$  is found. Also, if the latitude of the place be known,  $PV$ , the distance of the elevated pole from the vertex, or the complement of its distance from the horizon, that is, (by Prop. III.) the complement of latitude is known. Therefore in the spherical triangle  $PVS$ , two angles and one side are known; whence the length of  $SV$  may be determined. Take  $SV$  from  $EV$ , which is already known, and  $SE$ , the planet's parallax, will be found.

Or thus: observe when the planet whose parallax is to be found, and any fixed star in conjunction with it, cross the meridian at the same instant: observe the same planet and star after three hours, and remark how much sooner the planet reaches a line placed perpendicularly in the telescope than the star. As 24 hours is to this difference of time, so will 360 degrees be to the arc which subtends the angle of the parallax; whence the arc and angle will be known.

OR

## P R O P. CXXVIII.

Any parallax of a planet being given, to find any other parallax.

Plate 10.  
Fig. 17.

The parallactic angle  $AFC$  being given, it is required to find the angle  $AHC$ . Having measured the angle  $ZAL$ , let the angle  $ZAH$ , the apparent distance of the planet from the zenith, be also measured. Then, in the triangle  $CAF$ , the sine of the angle  $CAF$  is to the sine of the angle  $CFA$ , as the side  $CF$  is to the side  $AC$ . Again, in the triangle  $CAH$ , the sine of the angle  $CAH$  is to the sine of the angle  $CHA$ , as  $CH$  is to  $AC$ . But  $CH$  is equal to  $CF$ ; therefore the sine of the angle  $CAF$  is to the sine of the angle  $CFA$ , as the sine of the angle  $CAH$  is to the sine of the angle  $CHA$ : but the three first terms are known, therefore the fourth, namely, the angle  $CHA$ , may be found.

P R O P.



## P R O P. CXXIX.

At a given altitude of different planets, their diurnal parallaxes are inverfely as their distances from the center of the earth.

Let one planet be at  $f$ , where its altitude is  $fAp$ , and another at  $b$ , having an equal altitude  $bAp$ . If the planet  $f$  is viewed from  $A$  at the furface of the earth, the line of view is  $Afr$ , and  $r$  is its apparent place in the heavens: viewed from  $C$ , its apparent place would be  $t$ ; whence, its parallax (by Prop. CXXV.) is  $rt$ . In the fame manner it may fhewn, that  $rs$ , which is lefs than  $rt$ , is the parallax of the planet  $b$ . But (by Prop. CXXIV.) the parallax of each planet is proportional to the angle which  $AC$  would fubtend, if viewed from the planet: and, fince  $AC$  is given, and alfo the degree of obliquity at which it is viewed, the apparent length of  $AC$ , or the angle which  $AC$  would fubtend, at either planet, would be (by Book IV. Prop. LXIX.) inverfely as the planet's diftance from  $C$ . Therefore the parallaxes of thefe planets are inverfely as their diftances from the center of the earth.

Plate 10.  
Fig. 17.

## P R O P. CXXX.

The diurnal parallax of a planet in a vertical circle produces a parallax of declination, and alfo, if the planet is not in the meridian, of right afcenfion.

Let  $HQ$  be the horizon;  $EC$  an arc of the equator, which cuts the horizon at  $C$ ;  $P$  the pole of the equator;  $Z$  the zenith;  $ZV$  a vertical circle;  $F$ , the apparent place of a planet in the vertical circle  $ZV$ , as viewed from the furface of the earth, and  $I$  its apparent place, as viewed from the center: then (by Def. LXIII.)  $FI$  is the diurnal parallax in a vertical circle. When the apparent place is  $F$ ,  $PFA$  is a fecondary of the equator paffing through it, and when it is  $I$ ,  $PIB$  is the fecondary which paffes through it. Therefore  $AF$  is the declination of the planet when it appears at  $F$ , and  $BI$  its declination when it appears at  $I$ ; the difference of which,  $DI$ , is the change of apparent declination arifing from the different ftation of the fpectator, at the furface or center of the earth. When the apparent place is  $F$ , the diftance of  $A$  from the firft of Aries, is the right afcenfion; when it is  $I$ , the diftance of  $B$  from the firft of Aries, is the right afcenfion: for  $PFA$  and  $PIB$  are fecondaries of the equator paffing through the planet. The difference of right afcenfion, therefore, produced by the parallax  $FI$ , is  $AB$ . If the planet is in the meridian  $PZH$ , and if  $L$  be its apparent place, as viewed from the furface, and  $N$ , as viewed from the center of the earth,  $LN$  will be its diurnal parallax;  $LE$  its declination, as viewed from the furface;  $NE$  its declination, as viewed from the center; and  $NL$  its parallax of declination. But, becaufe  $PZH$  is a fecondary of the equator,

Plate 11.  
Fig. 1.

in



in whatever part of this vertical circle the planet appears, its right ascension will be the distance of the point E from the first of Aries, that is, the diurnal parallax, in this case, makes no parallax of right ascension.

## P R O P. CXXXI.

The diurnal parallax of a planet in a vertical circle produces a parallax of latitude, and also, if the vertical circle be not a secondary of the ecliptic, of longitude.

Plate 11.  
Fig. 1.

Let HQ be the horizon; P the pole of the ecliptic; EC an arc of the ecliptic, which cuts the horizon at C; and ZV a vertical circle: and this Proposition may be proved in the same manner as the last.

## P R O P. CXXXII.

The semidiameter of the earth is to the distance of any planet from the center of the earth, as the sine of the planet's parallax is to the sine of its apparent distance from the vertex.

Plate 10.  
Fig. 17.

If a planet is at F, and the spectator at A, where the line of view is AFL, the planet will appear at L, and ZAL will be the angle of its apparent distance from the vertex Z. Let the parallax IL, or the angle AFC proportional (by Prop. CXXIV.) to IL, be found. In the plane triangle ACF (the sides being to one another as the sines of the opposite angles) AC, the semidiameter of the earth, is to FC, the distance of the planet from the center of the earth, as the sine of the angle AFC the angle of the parallax, is to the sine of the angle FAC, or of its complement to two right angles ZAL, the angle of the planet's apparent distance from the vertex.

COR. When the horizontal parallax is taken, the semidiameter of the earth AC is to HC, the distance of the planet, as the sine of the horizontal parallax AHC is to the sine of HAC or radius.

## P R O P. CXXXIII.

To measure the distance of the moon from the earth.

Plate 10.  
Fig. 17.

Let H be the moon in the sensible horizon observed by a spectator at A, and C the center of the earth. In the triangle AHC, let the angle AHC, the moon's horizontal parallax, be found, by Prop. CXXVII. The angle HAC is a right angle, and AC the semidiameter of the earth is known to be 3985 miles. Hence, AC the sine of AHC is to

3985,



3985, as AH, taken as radius, to the number of miles in AH the moon's distance from the earth; the moon's mean distance is thus found to be 240,000 English miles.

SCHOL. The semidiameter of the earth is thus found: because the circumference of a circle is to its diameter nearly as 355 to 113; as 355 is to 113, so is 24930 miles, the known circumference of the earth, to its diameter 7970; the half of which is 3985.

## P R O P. CXXXIV.

To determine the relative distances of the inferior planets from the sun.

Let S be the sun, EHG the orbit of Venus, and LCM the orbit of Mercury. Let AXF be a tangent to the orbit of Venus, and let the elongation of Venus, that is, the angle XAS, be found by observation. Then as radius to the sine of the angle XAS, so is AS to XS or ES. In like manner, if the elongation of Mercury, or the angle CAS, be observed; as radius to the sine of CAS, so is AS to CS or LS. If AS, the sun's distance from the earth, be supposed to be divided into 1000 equal parts, the distance of Mercury will in this manner be found to be 387, and that of Venus 723.

Plate 9.  
Fig. 12.

## P R O P. CXXXV.

To determine the relative distances of the superior planets from the sun.

Let S be the sun, *nkg* the orbit of the earth, OPQ the orbit of Mars, and NKG a part of a great circle in the heavens, in which the planet appears to have a retrograde motion; let P be the place of Mars. Whilst the earth is passing in its orbit from *k* to *n*, Mars will appear to move from K to N. The angle of retrogradation KPN is then known by observation. To this the verticle angle *nPS* is equal. In the triangle *nSP*, the angle at *n* is a right angle; the angle *nPS* is the angle of retrogradation which is known, whence the other angle *nSP* is known, and the ratio of the sides of the triangle to each other is known: whence the ratio of *Sn* to *SP* is found. If the mean distance of the earth from the sun be called 1000, that of Mars will be found to be 1523, that of Jupiter 5201, and that of Saturn 9538.

Plate 10.  
Fig. 1.

## P R O P. CXXXVI.

To find the parallax of the sun by the transit of Venus.

Let FIG be the earth, L Venus, and S the sun. To an observer at I, Venus will appear just entered upon the sun's disc at C, and its apparent place in the heavens will be Q. But at the same instant to a spectator at G it will appear above the sun in the right

Plate 11.  
Fig. 2.

Q q



right line GLN. The angle NLQ, or BLG, is the horizontal parallax of Venus; and the angle BCG, or its equal TCQ, is the sun's horizontal parallax. Because the plane of Venus's orbit is inclined to the plane of the ecliptic, this planet, in its inferior conjunction, will commonly pass northward or southward of the sun; but, when the inferior conjunction happens at, or very near, one of the nodes, it will pass over the sun's disc. If, at the time of this transit, a spectator at I, and another at G (at the distance of 90 degrees, in longitude, from each other) observe the exact time of the total ingress of the planet, the difference, between the actual time of ingress, and the time at which the ingress would have happened without any parallax, being known, and (by Prop. XXVII.) converted into parts of a degree, the horizontal parallax of Venus, ILG, may be found. Or, if several different observations of the time of ingress be made at different places, after due allowance is made for difference of longitude and latitude at those places, the parallax of Venus at that time, and thence its horizontal parallax, will be accurately discovered.

If two observations of this angle be taken at the same time on opposite meridians, the errors attending the observations may serve to correct each other. For, suppose MO to be a part of the orbit of Venus, and V, *v*, *u*, the planet in different situations. The time in which Venus will pass through such an arc V*v* of its orbit, as when viewed from the earth subtends an equal angle with the diameter or chord CD of the sun being found, and also the arc V*v*; let a spectator on the earth's surface view the planet just entering within the sun's disc at C. If the earth remained at rest, the spectator would see the planet passing over the disc in the line CD, whilst in its path it describes V*v*: but because he is, in the mean time, by the earth's diurnal revolution, carried from A towards P, at the instant when he sees the planet passing off the sun's disc at D, the planet is advanced in its orbit to U. Consequently, the transit will be to this observer as much longer than the computed time, as the heliocentric arc VU is longer than V*v*. If another observation of the same kind be taken at the same time on the opposite meridian, where the spectator is carried in a direction contrary to the former, the duration of the transit will to him be as much shorter than the computed time, as in the other situation it was longer.

The horizontal parallax of Venus being thus found, the sun's horizontal parallax, on the day of the transit, may be discovered by Prop. XXIX. For, as the sun's proportional distance, at the transit, from the earth (taking 1000 for his mean distance) is to the proportional distance of Venus from the earth at that time, so is the horizontal parallax of Venus, ILG, to the horizontal parallax of the sun, ICG, on the day of the transit. Whence the sun's horizontal parallax, at the time of his mean distance from the earth, may be found: for (by Prop. XXIX.) as the sun's mean distance from the earth, is to his proportional distance at the transit, so is his horizontal parallax at that time to his mean horizontal parallax.

In this manner the sun's mean horizontal parallax has been found, from comparing the transits of Venus in 1761 and 1769, to be 8.65". or about  $8\frac{2}{3}$  seconds. See Philosophical Transactions, vol. LXII. p. 611, and Ferguson's Astronomy, Chap. XXIII.

P R O P.



## P R O P. CXXXVII.

To find the distance of the sun from the earth.

In the triangle AHC, suppose H to be the sun. As the sine of  $8\frac{2}{3}$  seconds, the horizontal parallax of the sun AHC, is to radius, so is the semidiameter of the earth AC, which is found by mensuration to be 3985 English miles, to the number of semidiameters of the earth contained in the distance of the sun from the earth. Hence the sun's distance from the earth is found to be about 95,173,100 English miles.

Plate 10.  
Fig. 17.

## P R O P. CXXXVIII.

To measure the distance of Mercury or Venus from the sun.

Let S be the sun, E the earth, and M Mercury or Venus. Measure the angle SEM, and observe accurately the time when this measure is taken. When Mercury has made one revolution, and arrives at the same point M, the earth will be in some other part of its orbit, as R; measure at that time, the angle SRM, and observe the time when the measure is taken.

Plate 11.  
Fig. 8.

By these two observations the time in which the earth passes from E to R is known: hence, as 1 year is to the time employed in passing from E to R, so are 360 degrees to the arc ER: whence the arc ER, and the angle ESR, are found. In the triangle ESR, the sides SE, SR, (the distance of the sun from the earth) being known, and also the contained angle RSE, let the angles at the base SER, SRE, and the base RE, be found. Then from the known angle SER take away the angle SEM, which is also known, there will remain MER; and from the known angle SRE take away the known angle SRM, there will remain MRE. The two angles MER, MRE, being thus found, the third angle RME is also known; and the side RE is known. Wherefore, the sine of the angle RME is to the side RE, as the sine of the angle MRE is to the side ME, or as the sine of the angle MER is to the side MR. In the triangle SRM, the sides RS, RM, being thus found, the sum of the two sides RS, RM, is to their difference, as the tangent of half the sum of the angles at the base RSM, RMS, is to the tangent of half their difference. To half the sum add half the difference, and the greater angle at M is found; and from half the sum take away half the difference, and the lesser angle at S is found. Whence, the sine of the angle at M is to the side RS, or the sine of the angle at S is to the side RM, as the sine of the angle at R is to the base SM, which is the distance required.



## P R O P. CXXXIX.

To measure the distance of Mars from the ~~earth~~ sun.

Plate II.  
Fig. 4.

Let S be the sun, E the earth, and M Mars. Measure the angle SEM: when Mars has made one revolution, observe the place of the earth in its orbit R, and measure the angle SRM. Having found as before the arc ER, and the angle ESR, in the triangle ESR, in which the two sides SE, SR, and the contained angle ESR, are known, let the angles at the base SER, SRE, and the base RE, be found. If from the angle SEM (which has been observed) be taken SER, there remains REM; and if from the angle SRM (which has been observed) be taken SRE, there remains ERM. Whence, in the triangle RME, the angles at R and E being found, the third angle is known: and the sine of the angle at M is to the side RE, as the sine of the angle at E is to the side RM. Wherefore, in the triangle SRM, the two sides of which, RS, RM, and the contained angle at R, are known; whence, as in the preceding Proposition, the two angles at the base S, M, and lastly the base SM, which is the distance required, may be found.

## P R O P. CXL.

To measure the distance of Jupiter or Saturn from the sun by their satellites.

Plate II.  
Fig. 5.

Let S be the sun, E the earth, and J Jupiter. First, observe the instant in which the Satellite R disappears behind Jupiter, and the instant in which it again appears: then, dividing the intermediate time into two equal parts; this will give the instant in which the earth E, Jupiter J, and the Satellite R, are in one right line EID. Next, observe the instant in which the Satellite disappears behind the shadow of Jupiter, and the instant in which it again appears; and divide the time between these instants into two equal parts to find the instant in which the Satellite is in the midst of the shadow, that is, in which the Sun, Jupiter and the Satellite form a right line SIR. Hence, the time taken up in passing from D to R, is known: whence, the time of the entire revolution of the Satellite is to 360 degrees, as the time employed in passing from D to R is to the arc DR. Thus the arc DR, and the angles RID, EIS, are found. Lastly, having taken an observation of the angle IES, the other angle ESI is found; and the side ES, the earth's distance from the sun, is known: whence, the sine of the angle EIS is to the side ES, as the sine of the angle IES is to the side IS, the distance required.

P R O P.



## P R O P. CXLI.

To measure the distance of any planet from the sun.

Because the real distances of the planets from the sun are as their proportional distances; as the proportional distance of the earth from the sun is to the proportional distance of any other planet from the sun, so is the real distance of the earth from the sun in miles, to the real distance of any other planet from the sun in miles.

Hence are found the distances of the planets from the sun in English miles. Mercury, 36,503,408; Venus, 68,209,408; Mars, 143,683,080; Jupiter, 490,452,000; Saturn, 899,561,100.

## P R O P. CXLII.

The horizontal parallax of any planet being given, to find its distance from the earth.

Let  $H$  be the planet, whose horizontal parallax  $AHC$  is known. The semidiameter of the earth  $AC$  being known, in the triangle  $CAH$  the sine of the angle  $AHC$  is to the side  $AC$ , as the sine of the angle  $HAC$  is to the side  $HC$ , the distance sought. Plate 10.  
Fig. 17.

## P R O P. CXLIII.

The distance of any planet being given, to measure its real magnitude.

Let  $A$  be the earth, and  $C$  the center of any planet; and let the distance  $CA$  be known. Suppose two right lines,  $AB$ ,  $AD$ , drawn tangents to the planet, the angles  $CBA$ ,  $CDA$ , are right angles; therefore the square of  $AC$  is equal to the two squares of  $AB$  and  $BC$  together. The same square of  $AC$  will also be equal to the two squares of  $AD$  and  $CD$ . And since the square of the radius  $CB$  is equal to the square of the radius  $CD$  (on account of the spherical figure of the planets) the square of the tangent  $AB$  is equal to the square of the tangent  $AD$ , and the tangent  $AB$  to the tangent  $AD$ . Hence the two triangles  $ABC$ ,  $ADC$ , are equal, and consequently, the angles  $BAC$ ,  $CAD$ , are equal. The angle  $BAD$  being measured by a micrometer, its half  $BAC$  is known; whence, in the triangle  $ABC$ , the sine of the angle at  $B$ , which is a right angle, is to the side  $AC$ , as the sine of the angle at  $A$  is to the side  $BC$ . The radius, and consequently the diameter of the planet, being thus found; because spheres are as the cubes of their diameters, its magnitude is known by finding the cube of its diameter. Plate 11.  
Fig. 6.

The



The diameter of the sun is found to be 878808 English miles; of Mercury, 3207; of Venus, 7649; of Mars, 4168; of Jupiter, 88724; of Saturn, 78648.

## P R O P. CXLIV.

To find the periodical time of a planet.

Because, whilst any planet is performing its revolution the earth is carried forward in its path, the planet, after one greatest elongation, must not only complete a revolution, but likewise the whole angular space which the earth described in that time, before it arrives again at the same elongation. Thus, before Venus can return to the same elongation, besides performing an entire revolution in its orbit (equal to 4 right angles) it must pass through as much more angular space, as the earth has done in the mean time. Hence, as the angular motion of Venus is to the angular motion of the earth in the time between the greatest elongation and its return, so is the periodical time of the earth to the periodical time of Venus. In this manner the periodical times of all the planets may be found. The periodical times of the planets are as follows.

|                                   | Days  | Hours | Minutes.         |
|-----------------------------------|-------|-------|------------------|
| Mercury revolves round the sun in | 87    | 23    | 15 $\frac{1}{2}$ |
| Venus - - - - -                   | 224   | 16    | 49 $\frac{1}{4}$ |
| Earth - - - - -                   | 365   | 6     | 9 $\frac{1}{4}$  |
| Mars - - - - -                    | 686   | 23    | 30 $\frac{3}{4}$ |
| Jupiter - - - - -                 | 4332  | 8     | 51 $\frac{1}{2}$ |
| Saturn - - - - -                  | 10761 | 14    | 36 $\frac{3}{4}$ |

COR. Because the squares of the periodical times of the planets are found to be as the cubes of their distances, the periodical times of any two planets being known, and the comparative or real distance of one of them from the sun being given, the distance of the other may from this proportion be found.

## P R O P. CXLV.

To find the mean velocities of the planets.

The periodical time of a planet being known, and also the diameter, and consequently the circumference of its orbit (for the diameter of a circle is to its circumference nearly as 113 to 355); its mean velocity, or the velocity with which it would move if its motion were uniform, may be thus found: as the whole periodical time of the planet is to an hour, so is the whole circumference of its orbit to the angular space passed over in



an hour. Thus it is found, that the mean horary velocity of the earth is 68,216.9 English miles. In like manner, the horary velocity of the other planets may easily be found.

## P R O P. CXLVI.

The planets revolve round their axes.

It is found by observation, that the earth revolves round its axis in 23 hours 56 minutes; Venus in 23 hours; Mars in 24 hours 40 minutes; and Jupiter in 9 hours 56 minutes. And though Mercury is so near the sun, and Saturn so remote from him, that no spots have been discovered in them, from whence their diurnal revolution might be certainly inferred, it is concluded from analogy, that these two planets have such a revolution as well as the rest.



B O O K V. P A R T II.

*Of the CAUSES of the CELESTIAL MOTIONS  
and of other PHENOMENA.*

C H A P. I.

*Of the Cause of the REVOLUTIONS of the HEAVENLY BODIES  
in their ORBITS.*

P R O P. CXLVII.

The moon is retained in its orbit by a force which impels it towards the center of the earth.

Since (by Book II. Prop. I.) the moon, or any other planet, being put into motion, would continually move on uniformly in a right line, there must be some force which draws it from its rectilinear path. Whatever this force is, since it is found by observation that the moon by a radius drawn to the earth's center describes equal areas in equal times, it follows (from Book II. Prop. LXXIII.) that it is impelled, by that force, towards the earth's center. The earth indeed is not at rest; but because both the moon and earth revolve round the sun, the motion of the moon with respect to the earth is the same as if the earth were at rest.

P R O P. CXLVIII.

The force which retains the moon in its orbit is, at different distances from the earth, inversely as the squares of those distances.

The moon's orbit being an ellipse, which has the earth in one of its foci, the force which retains it in its orbit, must (by Book II. Prop. LXXXI.) in different parts of the orbit be inversely as the squares of the distances from the earth.

P R O P.



## P R O P. CXLIX.

The moon is retained in its orbit by a force which carries it towards the earth with the same velocity with which a body, acted upon by gravitation at the distance of the moon, would fall towards the earth.

Let AER be the earth, PLV a part of the moon's orbit, LC an arc which the moon describes in its orbit in one minute of time. Since the moon describes its whole orbit in 27 days 7 hours 43 minutes, that is, in 39343 minutes, the length of the arc LC, which the moon describes in one minute, is the  $\frac{1}{39343}$  part of  $360^\circ$ : or  $33''$ . If the moon setting out from L, were not impelled towards the earth, it would move in the right line LB. Since therefore it moves in the arc LC, there must be a force impelling it towards the earth's center which draws it from the tangent LB, so that, at the end of 1 minute, when it is arrived at C, it will have departed from the tangent as far as BC, or LD: or, because the moon describes the diagonal LC in 1 minute, it would in the same time, by the projectile force, describe LB, and by the centripetal, LD. LD is then the space through which the centripetal force makes the moon fall towards the earth in 1 minute; and LD is the versed sine of the arc LC, which is an arc of  $33''$ . Therefore the force which impels the moon, would make it fall, in 1 minute of time, through the versed sine of an arc of  $33''$ . Because AER, the earth's circumference, is found to measure 123249600 Paris feet, its semidiameter AT will be about 19615800 such feet. Since therefore the mean distance of the moon from the earth is found to be 60 semidiameters of the earth, AT multiplied by 60 will give the length of LT a semidiameter of the moon's orbit, namely, 1176948000 feet. And as radius is to the versed sine of  $33''$ . so is LT to LD, or nearly as 1176948000 to  $15\frac{1}{12}$  Paris feet, which is nearly equal to  $16\frac{1}{2}$  English feet, or 1 pole. Therefore, if the moon were to fall towards the earth, the centripetal force which impels it towards the earth would make it fall 1 pole in the first minute of its descent. But because (by Prop. CXLVIII.) this centripetal force is inversely as the squares of the distances, a body which is at the distance of the moon, or 60 semidiameters of the earth, will be attracted by a force as much less than that at the surface, as the square of 60, or 3600, is greater than the square of 1, or 1; that is, the force at the surface being 1, it will be to the force at the distance of the moon, as 1 to  $\frac{1}{3600}$ , and the velocities will have the same ratio. But a body at the surface of the earth falls through 1 pole in a second of time, that is, (by Book II. Prop. XXVI.) through 3600 poles in a minute. Therefore, at the distance of the moon the body would fall through  $\frac{1}{3600}$  part of this length, that is, through 1 pole, in a minute. But it has been shewn that the moon, by its centripetal force, falls towards the earth 1 pole in a minute: therefore the moon is retained in its orbit by a force which

Plate 11.  
Fig. 7.

R r

moves



moves it with the same velocity with which a body, acted upon by gravitation, and removed to the distance of the moon from the earth, would fall towards the earth.

P R O P. CL.

The moon is retained in its orbit by the force of gravitation.

The force which retains the moon in its orbit agrees with gravitation in its direction (by Prop. CXLVII.) and in its degree of force (by Prop. CXLIX.); it may therefore be concluded to be the force of gravitation.

P R O P. CLI.

The primary planets are retained in their orbits by a force which impels them towards the sun.

It is found by observation, that each of them, as they revolve in their respective orbits, describe, by a radius drawn to the sun, equal areas in equal times. Therefore (by Book II. Prop. LXXIII.) the force which retains them in their orbit, impels them towards the sun.

P R O P. CLII.

The forces which retain the primary planets in their respective orbits are, at different distances from the sun, inversely as the squares of those distances.

It appears from observation, that the squares of the periodical times of the planets are as the cubes of their mean distances from the sun. For example, Saturn's periodical time being found to be to Jupiter's about as 30 to 12, and the distance of the former from the sun, to that of the latter nearly as 9 to 5; the squares of the times are 900 and 144; which are to one another nearly as 729 to 125, the cubes of the distances. This proportion takes place in all the primary planets: hence (by Book II. Prop. LXXIX.) the force by which they are retained in their orbits is inversely as the squares of the distances.

P R O P. CLIII.

The primary planets are retained in their orbits by the force of gravitation.

The moon having been shewn from the direction, and the law of its centripetal force, to be retained in its orbit by gravitation; since the primary planets are impelled towards the sun (by Prop. CLI.) as the moon is towards the earth, and since their centripetal force



force acts with respect to the sun by the same law, by which the force which retains the moon in its orbit acts with respect to the earth, namely, that this force is inversely as the square of the distance of the planet from the sun; it may be concluded, as in the case of the moon, that they are retained in their orbits by the force of gravitation.

This follows likewise from their moving in elliptical orbits, since it has been proved (Book II. Prop. LXXXI.) that bodies revolving in such orbits have their centripetal forces inversely as the squares of their distances from the center about which they revolve.

## P R O P. CLIV.

The satellites of Jupiter and Saturn are retained in their respective orbits by the force of gravity.

They are observed to describe equal areas round the respective primaries in equal times, and, consequently (by Book II. Prop. LXXIII.) are impelled towards them; and the forces which retain them in their orbits, are at different distances inversely as the squares of those distances (by Book II. Prop. LXXXI.) because it has been observed that the squares of their periodical times are as the cubes of their distances from their respective centers. Therefore the force which retains the satellites of Jupiter and Saturn in their orbits, act in the same manner, and by the same law, as the force which retains the moon in its orbit acts with respect to the earth. But all effects of the same sort are, without proof of the contrary, to be considered as produced by the same cause. Therefore the power which retains the satellites in their orbits is gravitation.

## P R O P. CLV.

The sun and any planet revolve round a common center of gravity, which remains at rest.

Let S be the sun, and P any planet, mutually attracting each other. If neither of the two bodies revolved in any orbit, they would move towards each other, and would meet at C their common center of gravity; and during the approach of these two bodies, C their common center of gravity would be at rest, by Book II. Prop. LI. But if the body P has a projectile force given to it in the direction Pt, and if this projectile force and its gravitation towards S make it describe an orbit round S, (by Book II. Prop. LXVIII.) such a projectile force will prevent the body P from approaching to S, though it gravitates towards S. But if S has not as great a projectile force given to it at the same time in the opposite direction Ss, then because S continues to gravitate towards P, and there is no force which can prevent its approaching to P, it follows that S will approach to P, or as P revolves round S the mutual gravitation of these two bodies will diminish the distance SP. Now it appears from Book II. Prop. LI. that C the common center of

Plate II.  
Fig. a.



gravity always divides this distance  $SP$  in the inverse ratio of  $S$  to  $P$ , or that  $SC$  is always as much less than  $PC$ , as the quantity of matter in  $S$  is greater than the quantity of matter in  $P$ : consequently, since the quantity of matter in  $S$  and in  $P$  is always the same,  $SC$  and  $PC$  have always the same ratio to one another. But as  $S$  approaches to  $P$ ,  $SC$  decreases. Therefore  $PC$  must decrease in the same ratio. But  $PC$  can decrease no otherwise than either by the approach of  $P$  to  $C$ , or by the approach of  $C$  to  $P$ . But the projectile force prevents  $P$  from approaching to  $C$ . Therefore  $C$  must approach to  $P$ . Thus it appears that, if  $P$  has a projectile force given to it, and is made to revolve, unless  $S$  has an equal projectile force given to it at the same time, the mutual gravitation of these two bodies towards one another will put  $C$ , which is their common center of gravity, in motion: contrary to Book II. Prop. LI. Cor. Therefore as the planet  $P$  begins to move in the direction  $Pt$ , the sun  $S$  will likewise begin to move in the direction  $Ss$ ; and  $C$  their common center of gravity will continue at rest. And as they tend mutually towards each other, or towards  $C$  their common center of gravity, their motions will not continue to be performed in right lines, but the planet  $P$  will revolve round  $C$  in an orbit of which  $PR$  is a part, and the sun  $S$  will revolve round  $C$  in an orbit of which  $SQ$  is a part.

## P R O P. CLVI.

The sun and any planet, whilst they mutually gravitate towards each other, describe similar figures round their common center of gravity, and round each other.

Plate 11.  
Fig. 2.

Let  $S$  be the sun,  $P$  the earth, or any other planet, and  $C$  their common center of gravity, about which (by the last Prop.) they revolve. To a spectator at  $P$ , who imagines the planet to be at rest, the sun will appear to revolve about  $P$ , and the reverse at  $S$ . Because the common center of gravity of the sun  $S$ , and any planet  $P$ , is always in a right line drawn from the sun to the planet, if the planet moves through any small space from  $P$  to  $p$ , the line  $pC$  continued must pass through the sun; and consequently the sun must have moved from  $S$  to  $s$ . Thus  $Pp$ ,  $Ss$ , are arcs described by the planet and sun in their respective orbits in the same time, and  $PCp$ ,  $SCs$ , are areas described in the same time by the radii  $CS$ ,  $CP$ . And, because the vertical angles at  $C$  are equal, and that  $SC$  is to  $PC$  (as  $sC$  to  $pC$ , for  $SP$ ,  $sp$ , are both divided in  $C$  in the inverse ratio of the quantities of matter in  $P$  and  $S$ ) the areas  $PCp$ ,  $SCs$ , are similar. In like manner, any other parts of the two orbits described in the same time, may be shewn to be similar; consequently, the whole orbits are similar.

Again, when  $P$  has completed its revolution round  $C$ , or  $\frac{1}{2}$  or  $\frac{1}{4}$  of its orbit, it will appear to a spectator at  $S$ , to whom  $S$  seems at rest, to have completed its orbit, or  $\frac{1}{2}$  or  $\frac{1}{4}$  of its orbit round  $S$ . And universally, the angular motion of the planet  $P$  about  $C$ , in any given time, will be equal to its apparent angular motion about  $S$ , considered as

at



at rest by a spectator at  $S$ . If therefore the planet  $P$  in any given time has moved from  $P$  to  $p$ , in which (by the last Prop.) the sun  $S$  has moved from  $S$  to  $s$ , the angle  $PCp$ , which is the measure of the planet's angular motion about  $C$ , will be equal to the apparent angular motion round  $S$ . Let  $St$  be taken equal to  $sp$ , and make the angle  $PS t$  equal to the angle  $PCp$ ;  $P$  will by a radius drawn to  $S$  apparently describe the area  $PS t$ , whilst, by a radius drawn to  $C$ , it is describing the area  $PCp$ . Now, because (as was before shewn)  $SC$  is to  $PC$ , as  $sC$  to  $pC$ , (El. V. 18.)  $SC + CP$ , or  $SP$  is to  $PC$ , as  $sC + Cp$ , or  $sp$  is to  $pC$ ; and  $sp$  is equal to  $St$ : therefore  $PS$  is to  $PC$ , as  $St$  is to  $pC$ . Consequently, the two figures  $PCp$ ,  $PS t$ , are similar. In like manner it may be shewn, that any other part, described in any given time, of the orbit the planet appears to move in round the sun considered as at rest, will be similar to other parts, described in the same time, of the orbit in which the planet moves round the common center of gravity of the sun and the planet: therefore the whole orbits are similar. And since the orbits which the sun, and the planet, describe about their common center have been proved to be similar, it follows, that the orbit which any planet appears to describe round the sun considered as at rest, is similar to the orbit which the sun in the mean time describes round the common center of gravity.

In like manner it might be proved, that the orbit which the sun  $S$  appears to describe round the planet  $P$  considered as at rest, is similar to either of the orbits which the planet and sun describe about their common center of gravity.

COR. If the sun's apparent motion, seen from the earth, is an ellipse, having the earth in one of its foci, the earth's apparent motion, seen from the sun, will be in a similar ellipse, having the sun in one of its foci: and if the sun and earth mutually gravitate towards each other, they describe similar elliptic orbits about their common center.

#### P R O P. CLVII.

The common center of gravity of the sun and all the planets is at rest, and is the center of the solar system.

Since, from the mutual gravitation of the sun and any one planet, they will revolve about their common center (by Prop. CLV.) the same must hold good with respect to the sun and all the planets. Consequently, there must be some one point in the solar system which is its center of gravity, and is at rest.



## C H A P. II.

*Of the LUNAR IRREGULARITIES.*

## P R O P. CLVIII.

The nearer the moon is to its syzygies, the greater is its velocity; and the nearer it is to its quadratures, the slower it moves.

Plate 11.  
Fig. 8.

Let S represent the sun, T the earth, and LMNO the orbit of the moon; let the moon be in one of its quadratures at L, and let the lines LS and TS be drawn. It is obvious, that the tendency which the moon has towards the sun is along the line LS, and that which the earth has, is along the line TS: let then the former of these be resolved into two others, the one along LA parallel and equal to TS, the other from L to T, along the line LT. The former of these tendencies being parallel and equal to that by which the earth tends along the line TS, alters not the situation of the two bodies L and T with respect to each other, that is, it disturbs not the motion of the body L; but the other along LT increases its tendency towards T. And this increase will be to the tendency the moon has to A, which is the same which the earth has to S, as the distance LT to LA, or TS: that is, the gravity of the moon towards the earth in the quadratures is augmented by the action of the sun; and that augmentation is to the tendency which the earth has to the sun, as the length of the line LT, or the distance of the moon from the earth, to TS the distance of the earth from the sun.

Hence the greater the moon's distance is from the earth, the distance of the sun remaining the same, the greater will this increase of the moon's gravity towards the earth be. But if the distance of the moon from the earth remains the same, and the distance of the sun be augmented, this additional increase will be the less in the ratio of the cube of that distance. For, if TS be increased while LT remains the same, LT will be so much the less with respect to TS, that is, the increase will be diminished in the ratio of the sun's distance: but when TS the distance of the sun is increased, the absolute force of the sun, and with it the above-mentioned increase, will be diminished also in proportion to the square of that distance; consequently, taking in both considerations, it will upon the whole be diminished in the ratio of the cube of that distance.

Let now the moon be in one of its syzygies at M, then will the tendency it has to the sun, more than that which the earth has, which is farther off at T, be to that which the earth



earth has, as the difference of the squares of SM and ST is to the square of SM: but the difference between the squares of SM and ST has nearly the ratio to the square of SM, which twice MT, that is MO, has to SM; because the difference between the squares of two numbers whose difference is very small with respect to either of them (as the difference between SM and ST is with respect to the distance of S) has little more than double the ratio to the square of the lesser number, that the difference between the numbers themselves has to the lesser number. The tendency therefore which the moon, when at M, has to the sun, more than that which the earth has, is to that which the earth has, nearly as MO, or twice TL, to SM, or because of the sun's great distance, as twice LT to TS. Her tendency therefore to the earth is now diminished in that ratio: but as was shewn above, it was augmented in the quadratures in the ratio only of LT to TS. The diminution here is therefore nearly double of the augmentation there. And whereas that augmentation, when the distance of the sun remains the same, was shewn to increase with the distance of the moon; but when the distance of the moon remains the same, to decrease with the cube of the sun's distance; this diminution being always nearly double of that, will do the same.

When the moon is in the other syzygy at O, it is attracted towards the sun less than the earth is by the difference of the squares of SO and ST; which, in effect, is the same thing as if the earth were not attracted at all towards S, and the moon were attracted the contrary way; so that its tendency to the earth is here also diminished, as well as when it was at M, and almost in the same degree; for on account of the sun's great distance, the difference between the squares of SO and ST is nearly the same as between ST and SM.

Or thus: the annual course of the moon round the sun being performed in the same time that the earth's is, it ought to be retained in that course by the same force that the earth is; whereas when the moon comes to M, the action of the sun upon it is greater than it is upon the earth, by the difference of the squares of SM and ST; and when the moon is at O, it is less than it is upon the earth by the difference between the squares of ST and SO: so that in the former case the moon is drawn too much towards the sun, and in the latter too little; and therefore in both cases its tendency towards the earth is diminished, and almost in the same degree; because, as was observed above, the difference of the above-mentioned squares is nearly the same in either case.

Next, let the moon be in a point of her orbit between the quadrature and the syzygy. Then being nearer the sun than the earth is, she will be attracted with a stronger force: let it be expressed by  $IS$  produced to  $D$ , till  $ID$  be of such length, that  $TS$  being put to express the action of the sun upon the earth,  $ID$  may express the stronger force of the sun upon the moon: and let  $ID$  be resolved into two others, one of which let be  $la$  equal and parallel to  $TS$ , then will the other be  $aD$ , or its equal and parallel  $IG$ . This  $IG$  is the only disturbing force upon the moon at  $L$ , the other  $La$  being parallel to  $TS$ , affects the moon just as the sun does the earth; and thus alters not their situations



Plate II.  
Fig. 9.

situations with respect to each other. Let then this figure with the line LG be transferred to fig. 9. This force LG may be resolved into LI and LH, the one a tangent to the orbit of the moon, and the other a perpendicular thereto: the former accelerates the motion of the moon when going from the quadrature at O to the syzygy at B; and will retard it when going from B to R. The other, when H falls upon TL produced, as in this figure, diminishes the tendency of the moon towards the earth; and when it falls between L and T, it augments it.

Thus the nearer the moon is to its syzygies, the greater will be its velocity; and the nearer it is to the quadratures, the slower it will move; because one of the forces into which LG is resolvable, accelerates its motion from the quadratures to the syzygies; and retards it as much from thence to the quadratures.

### P R O P. CLIX.

The moon describes equal areas in equal times only at the syzygies and quadratures, and deviates from this law the farthest in the octants.

Plate II.  
Fig. 9.

The disturbing force being resolved into two others, one of them, at the quadratures or syzygies, will be found to point from or towards T the center of the earth directly, and therefore will not hinder the moon from describing equal areas in equal times; the other likewise, in those places will be found to tend towards the center of the sun, and therefore neither of them will prevent the moon there from describing equal areas in equal times, that is, will not at the quadratures disturb the moon's motion at all.

But when the moon is in the octants, as at L, this force being resolved into two others, one of them, as LH, will point directly to or from the center of the earth, and therefore will increase or diminish the moon's tendency towards the earth, but not hinder it from describing equal areas in equal time. But the other, as LI, or HG, points neither towards the center of the earth, nor sun, and therefore, in the octants, prevents its describing equal areas in equal times. But this being the mid-way between the quadrature and the syzygy, in both which places this disturbing force doth not prevent the moon from describing equal areas in equal times, it follows, that at the octants, this disturbing force will be greatest of all.

### P R O P. CLX.

The orbit of the moon is more curved in the quadratures, and less in the syzygies, than it would be if it were only attracted by the earth.

For its motion (by Prop. CLVIII) being accelerated during its progress from the quadratures to the syzygies, in the syzygies its motion will be quicker than it ought otherwise



otherwise to be, and therefore its centripetal force less than it would otherwise be. It will therefore at the syzygies describe the portion of a larger curve, which consequently will be less curved than a smaller. On the other hand, while the moon passes from the syzygies to the quadratures, its motion is continually retarded, and therefore, at the quadratures, its motion will be slower than it would otherwise be. At the quadratures therefore the moon will describe the portion of a lesser curve, which therefore will be more curved than a larger curve.

P R O P. CLXI.

When the earth is in its perihelion, the periodical time of the moon will be the greatest; when the earth is in its aphelion, the periodical time of the moon will be the least.

Since the irregularities explained in the three preceding Propositions proceed from the action of the sun, it follows, that where the action of the sun is greatest, the irregularities arising from it will be greatest too. But the nearer the earth is to the sun, the greater will be the action of the sun upon the moon; and the more the moon tends towards the sun, the less will it tend towards the earth. When therefore the earth is at the perihelion A, and consequently at its least distance from the sun S, the action of the sun upon the moon will be greatest, and destroy more of its tendency towards the earth than at any other distance, as SE, SC, SD, &c. Therefore when the earth is at the perihelion A, the moon will describe a larger orbit about the earth, than when the earth is at any other distance from the sun, and consequently, her periodical time will then be the longest.

Plate 11.  
Fig. 10.

But the earth is at its perihelion in the winter, and consequently, then the moon will describe the outermost circle about the earth, and her periodical time will be the longest: which agrees with observation. For the same reason, when the earth is at its aphelion B, the tendency of the moon towards the earth will be the greatest, and consequently, her periodical time the least. And in this case, which will be in the summer, it will describe the innermost circle about the earth.

P R O P. CLXII.

The line of the moon's *apsis* goes forward when the moon is in syzygy, and backwards when it is in quadrature; but it goes farther forwards than backwards each time, so that at length it performs a revolution according to the order of the signs.

Since the moon describes an elliptical orbit CEDF about the earth S, placed in one of its *foci*, and since its centripetal force towards the earth, by means of the action of the

Plate 11.  
Plate 10.

S s

sun,



sun (by Prop. CLVIII.) is continually increasing, or decreasing, but not equably, that is, sometimes less, and sometimes more, than in the inverse duplicate ratio of the distance of the moon from the earth, therefore, the line of the moon's *apsis* AB will be continually going backwards, or forwards; that is, the axis AB will not always lie in that situation, but go backwards into the situation CD, or forwards into the situation EF. Since however, taking one whole revolution of the moon about the earth, the action of the sun more diminishes the tendency of the moon towards the earth than it augments it, therefore the motion of the *apses* forwards, exceeds their motion backwards. Upon the whole, therefore, the *apses* of the moon's orbit go forwards, or according to the order of the signs. Their revolution is completed in about 9 years.

## P R O P. CLXIII.

The excentricity of the moon's orbit is varied in every revolution of the moon, and is greatest when the moon is in syzygy, and least when it is in quadrature: and the orbit is most of all excentric when the line of the *apsis* is in the syzygies, and least of all excentric when this line is in the quadratures.

Plate 11.  
Fig. 10.

Because the moon describes an excentric orbit CEDB about the earth S, and the action of the sun upon it sometimes increases its tendency towards the earth, and sometimes diminishes it, that is, makes its gravity towards the earth increase or decrease too fast; if, while the moon ascends from her lower *apsis* A, its gravity towards the earth decreases too fast, instead of describing the path DBF, and coming to the higher *apsis* at B, it will run out into a curve beyond DBF, that is, the orbit will become more excentric, or farther from a circle. On the other hand, if the moon is passing from her higher *apsis* B, to her lower A, and its gravity towards the earth, by the action of the sun, increases too fast, it will approach nearer to the earth than the curve CAE, and describe a curve within CAE, or a portion of an orbit less excentric, or nearer to a circle, than CEDF. And if we compare several revolutions of the moon together, we shall find, that when the line of the *apsis* is in the syzygies, the excentricity will be the greatest of all, because in that situation, the difference between the tendency which the moon has to the earth in one of the *apses*, and that which it has in the opposite one, is the greatest of all; whereas, when the line of the *apsis* is in the quadrature, this difference is the least, and therefore the lunar excentricity will be so too.

P R O P.



## P R O P. CLXIV.

The line of the nodes moves backwards, but not uniformly; when it is in the fyzygies it stands still, and moves fastest in the quadratures.

When the line of the nodes is in the fyzygies, as CD, the plane of the moon's orbit passes through the center of the sun S, as well as through that of the earth E; whence, the disturbing force acting in the direction of the line of the nodes, and consequently in the plane of the lunar orbit, the moon is not drawn out of the plane of its orbit by the sun. But when the line of the nodes is in any other situation, and the moon not in one of the nodes, it is continually drawn out of the plane of its own orbit, on that side on which the sun lies. For instance, if the plane of its orbit CGDF produced passes above the sun, the sun draws it downwards; if, on the contrary, the plane of its orbit produced passes below the sun, it draws it upwards. Hence it follows, that when the line of the nodes is not in the fyzygies, and the moon having passed either of the nodes, has got out of the plane of the ecliptic ACBD on either side of it, the action of the sun occasions the moon to return back to the plane of the ecliptic sooner than it otherwise would do. But where the moon enters that plane, there is the next node; so that each node does, as it were, come towards the moon: and the nearer the line of the nodes is to the quadratures, the greater is this effect, because in that case, the sun is the farthest of all from the plane of the lunar orbit produced. So that the line of the nodes goes backwards the fastest of all, when it is in the quadratures; and not at all in the fyzygies.

Plate II.  
Fig. 11.

## P R O P. CLXV.

The inclination of the lunar orbit is liable to change, and is greatest when the nodes are in the fyzygies, and least when they are in the quadratures.

When the nodes are in the quadratures A, B, and the moon in its orbit AGBF has passed A, and is approaching the fyzygy which is next to the sun; the action of the sun upon the moon prevents its ascending so high, that is, departing so far from the plane of the ecliptic ADBC; whence the inclination of its orbit to the ecliptic will become less, and it will come to conjunction with the sun at H, making an angle with the ecliptic HAD, less than GAD. As the moon goes on to the next quadrature B, the action of the sun upon the moon, in its descent towards the node, hastens its descent, and thus, bringing it down to the ecliptic at K sooner than it would otherwise arrive there, increases the inclination of the plane of its orbit as much as it was diminished in ascending from

Plate II.  
Fig. 12.



A to H. And for the same reason, while the moon passes from B to the opposite syzygy F, the action of the sun decreases the inclination of its orbit, and increases it again on its passage from thence to A the next quadrature.

When the nodes are in the syzygies, C, D, the plane of the moon's orbit produced, passes through the center of the sun; and consequently, not being affected by the action of the sun, its inclination is neither increased nor diminished.

But while the nodes are passing from the syzygies C, D, to the quadratures A, B, the inclination of the moon's orbit is diminished in every revolution of the moon; and while they are passing from thence to the syzygies, it is continually increasing. Suppose the nodes in the octants at O and L, and the plane of AGBF, the orbit of the moon, so inclined to ADBC, the ecliptic, that if produced, it will pass above the sun S. When the moon is nearer the sun than the earth is, it is attracted towards the sun more than the earth is; and when farther off, the earth is attracted more than the moon is, that is, the moon is as it were attracted the other way. Hence, whilst the moon is *ascending* from the ecliptic in passing from O to P, the disturbing force being towards S, and the orbit above S, the moon will not rise so high as P, and the inclination of its orbit will be diminished while it is passing over 90 degrees from the node O. In going from a point below P to the next quadrature B, which is 45 degrees, the disturbing force being still towards S, because the moon is as yet nearer the sun than the earth is, and the moon now *descending* towards the ecliptic, the attraction of the sun will hasten its descent, and therefore cause it to move in a plane which will make with the plane of the ecliptic a larger angle than before; that is, in passing from P to B the inclination of the orbit is increased. But when the moon has passed B, and is moving towards L, the disturbing force acting, in the plane of the ecliptic, *from* the sun, and the moon still *descending* towards the ecliptic, the disturbing force, attracting the moon upwards, will retard its descent to the ecliptic, and cause it to move in a plane which will make a less angle with the plane of the ecliptic than before; that is, while it is passing from B to the node L, the inclination of its orbit is diminished. Thus, while the moon passes from O to L, the inclination of its orbit is diminished during three fourths of the passage. In like manner, while the moon is *ascending* from L to I, because the disturbing force acts *from* the sun, the inclination of its orbit is diminished; and while it is *descending* from I to A, the disturbing force still acting from the sun, the inclination is increased. But while it is still descending from A to O, because the disturbing force acts *towards* the sun, the inclination is diminished. Add to this, that while the moon passes from O to P, and from L to I, the disturbing force is much greater than when it is passing from P to L, and from I to O, because the difference between the distances of the moon and of the earth from the sun is greater in the former case than in the latter. On the whole therefore, while the nodes are between A and D, B and F, that is, while they are passing from syzygy to quadrature, the inclination of the lunar orbit is diminished; for, though the nodes have been supposed equally distant from the quadrature and syzygy, it is obvious  
that



that the like effects must happen, though different in degree, when they are nearer to the one than the other.

Next, let the nodes be in the octants I, P, between A and F, and B and G. While the moon is *ascending* from the node I towards the quadrature A, the disturbing force acting *from* the sun, it will be drawn upwards, and the inclination of its orbit will be hereby increased. In *ascending* from A to O, the disturbing force acting *towards* the sun, its ascent will be diminished, or the inclination of its orbit lessened: but in *descending* from O to the node P, the disturbing force still acting towards the sun, it will be drawn downwards, and consequently, the inclination of its orbit will be increased. Thus, during one whole revolution of the moon in this position of its nodes, the inclination of its orbit will be increased through three fourths of its passage. And this will be true, as in the other case, when the nodes are not in the octants. Also, for the reason mentioned in the other case, the force which increases the inclination of the orbit is, while it acts, superior to that which diminishes it. While the nodes therefore are passing from the quadratures to the syzygies, the inclination of the moon's orbit is increasing. From all which it is manifest, that the inclination of the lunar orbit is the least when the line of the nodes is in quadrature, and the moon in syzygy, and greatest when the line of the nodes is in syzygy.

#### P R O P. CLXVI.

The nodes of the moon are at rest, when the line of the nodes is in syzygy; they move *in antecedentia*, or from east to west, when the line of the nodes is in quadrature; and also when it is between quadrature and syzygy; but their regress in one revolution is, in this case, less than when the line of the nodes is in quadrature.

When the line of the nodes is in syzygy, because the disturbing force acts in the plane of the moon's orbit, it cannot change the inclination of that plane to the ecliptic; whence, the common intersection of the two planes, or the line of the nodes, is immovable. If, whilst the line of the nodes is in AB, the moon is passing from A through G to B, being constantly drawn towards the plane of the ecliptic by the disturbing force, it will come to the plane sooner than it would have done if no such force had acted upon it, that is, before it has described  $180^\circ$ . or is arrived at B.

Plate III.  
Fig. 12.

In like manner, while the moon is passing from B to A, through F, being drawn towards the plane of the ecliptic by the disturbing force, it will cross the ecliptic sooner than it would otherwise have done, that is, before it arrives at A. Consequently, the nodes will have changed their places, and moved in a contrary direction to the moon. In any other position of the line of the nodes, the disturbing force will, for the same reason, cause the line of the nodes to move *in antecedentia*, though in a less degree: be-  
cause,



cause, whilst the moon is describing the greater part of its orbit, it is drawn by the disturbing force (as was shewn in the last Prop.) towards the ecliptic, and consequently is made to cross the ecliptic sooner than it would otherwise have done, that is, the nodes are, on the whole, in one revolution of the moon, made to move in a direction contrary to that of the moon: but this regrefs is less than when the line of the nodes is in quadrature, because, during part of the revolution in this oblique position of the line of the nodes, the nodes move *in consequentia*, or in the same direction with the moon, namely, whilst the disturbing force (as was shewn in the last Prop.) draws the moon from the plane of the ecliptic; whereas, when the line of the nodes is in quadrature, they move *in antecedentia* during the whole revolution.

SCHOL. 1. The nodes perform one revolution, or pass through every part of the ecliptic, in about 19 years.

SCHOL. 2. All the irregularities of the moon are greater when the earth is in its perihelion, than when it is in its aphelion, because the effect of the sun's action, whereby they are produced, is inversely as the cube of its distance from the earth. They are also greater when the moon is in conjunction with the sun, than in opposition, for the same reason; for the earth and moon, taken together, are nearer the sun in the former situation of the moon, than they are in the latter.



## C H A P. III.

*Of the SPHEROIDICAL FORM of the EARTH.*

## P R O P. CLXVII.

In the daily revolution of the earth round its axis, the centrifugal force diminishes the weight of bodies more <sup>at the equator</sup> than in any other place on the surface of the earth, in the duplicate ratio of the semidiameter to the cosine of the latitude of the place.

Let  $PEpe$  be the earth,  $Pp$  the axis,  $Ee$  the equator. As the earth revolves upon its axis, every place on its surface, except the two poles, describes a circle, the plane of which is perpendicular to the axis, and the radius of which is the distance of that place from the axis. Thus, a body placed at  $A$  will in one revolution of the earth describe a circle, the semidiameter of which will be  $AB$ , which with the plane in which it lies will be perpendicular to the axis  $Pp$ . In like manner,  $CE$  is the semidiameter of a circle described by the revolution of a place in the equator. But  $CE$  is the semidiameter of the earth, and  $AB$  is the cosine of latitude of the place  $A$ ; for  $AB$  is the sine of  $AP$ , the complement of  $AE$ , which is the latitude of the place. And a body at  $E$ , revolving in a circle whose radius is  $CE$ , performs its revolution in the same time with a body at  $A$ , revolving in a circle whose radius is  $AB$ . But where the periodical times are equal, the centrifugal forces are as the radii, by Book II. Prop. LXXV. Whence the body at  $E$  has its centrifugal force as much greater than the body at  $A$ , as the radius  $CE$  is greater than the radius  $AB$ ; and universally, the centrifugal force at the equator is to the centrifugal force at any other place on the surface of the earth, as the semidiameter of the earth to the cosine of the latitude of the place. And since it is manifest that the gravity must be diminished as much as the centrifugal force is increased, the gravity of a body at the equator is as much less than that of a body at any other place on the earth, as the semidiameter of the earth is greater than the cosine of the latitude of the place.

Moreover, if the centrifugal forces at  $E$  and  $A$  were equal, they would diminish the weights of bodies unequally, on account of the different directions in which these forces act. The centrifugal force at  $A$ , acting obliquely upon the force of gravitation towards  $C$ , can only diminish this force by such a part of its action as is opposite to the direction of gravitation, that is, resolving  $Ab$  which may express the centrifugal force at  $A$  into

$Aa$ ,

Plate 11.  
Fig. 13.



$Aa$ ,  $ab$ , the part of the centrifugal force which will act to diminish the gravity of the body at  $A$ , will be to the whole centrifugal force at  $A$ , as  $Aa$  to  $Ab$ . Whereas at  $E$ , the whole centrifugal force, acting in direct opposition to the force of gravitation, will operate to diminish the weight of a body at  $E$ . Hence the force which acts to diminish the weight of a body, that is, the diminution at  $E$  is to the same at  $A$ , as the whole centrifugal force  $Ab$  to the part  $Aa$ . But  $Ab$  is to  $Aa$ , as  $CE$  to  $BA$ : for, the triangles  $Aab$  and  $ABC$  being similar,  $Ab$  is to  $Aa$ , as  $AC$  or  $EC$  to  $BA$ . Therefore, from the different directions in which the centrifugal forces act at  $E$  and  $A$ , the weight at  $E$  is as much more diminished than at  $A$ , as  $EC$  the semidiameter is greater than  $AB$  the cosine of the latitude of the place  $A$ .

The centrifugal force then diminishes the force of gravitation in the ratio of  $EC$  to  $AB$ , both because the centripetal force at  $E$  is greater than at  $A$ , and because it acts directly at  $E$ , but obliquely at  $A$ . Therefore the centrifugal force diminishes the weight of a body at  $E$ , more than at  $A$ , in the duplicate ratio of  $CE$  to  $BA$ , that is, as much more at  $E$  than at  $A$ , as the square of  $CE$  is greater than the square of  $BA$ .

SCHOL. It is found by calculation from this Prop. that gravity at the equator is diminished by the centrifugal force in the ratio of 288 to 289.

### P R O P. CLXVIII.

The earth is an oblate spheroid, elevated at the equator, and depressed at the poles.

It has been found by observation, that a <sup>shorter</sup> ~~longer~~ pendulum is required to vibrate seconds at the equator than at the poles; but (from Book II. Prop. XLIII. and XLIV.) the lengths of pendulums vibrating in the same time are as the gravities at the places where they vibrate; therefore the gravity at the poles is greater than at the equator. And it has been found that this difference of gravity is so much greater than would arise from the centrifugal force alone, that the ratio of the equatorial diameter of the earth to the polar diameter, must be as 212 to 211.

COR. 1. The degrees of latitude upon the earth's surface are longer at the poles than at the equator. For an arc of a meridian near the poles is less curved than near the equator, that is, it is an arc of a larger circle; whence a degree measured upon that arc must be greater than upon an arc of the same meridian at the equator.

COR. 2. The tendency of a heavy body, on any part of the surface of the earth between the poles and the equator, is not directly towards the center, but towards some point between the center and the equator.

SCHOL. The point towards which a body in any given place will tend may be determined.

For



For (by Prop. CLXVII.) as radius EC is to the cosine of latitude of the place AB, so is the centrifugal force at E to the centrifugal force at A in the direction Ab. Produce, therefore, the line BA to b, till Ab has the same ratio to AC, as the quantity last found has to gravity upon the surface of the earth. Complete the parallelogram AbCc; the point sought will be c, and the tendency of the body will be along Ac; thus suppose the latitude of the place  $51^{\circ}.46'$ . The centrifugal force at the equator is found to be to that of gravity, as 1 to 289: hence, as radius is to the cosine of  $51^{\circ}.46'$ . so is 1 to ,618, which is the centrifugal force at A. Consequently, the centrifugal force at A is to the force of gravity, as ,618 to 289: therefore, by the construction, Ab or Cc, is to AC in that ratio. The ratio of AC to Cc being thus found, as AC is to Cc, or as 289 is to ,618, so will the sine of the angle of latitude ACc, or  $51^{\circ}.46'$ . be to  $5^{\circ}$ . nearly, which is the angle required, measuring the deviation of the line of direction of falling bodies at the given latitude from a line drawn to the center of the earth.

Plate XX.  
Fig. 13.



## C H A P. IV.

*Of the PRECESSION of the EQUINOXES.*

DEF. LXIV. A *Periodical Year*, is the time in which the sun completes its revolution through the ecliptic.

DEF. LXV. A *Tropical Year*, is the time in which the sun completes its revolution setting out from any solstitial or equinoctial point, and returning to the same.

## P R O P. CLXIX.

The equinoctial points move *in antecedentia*, or go backwards from east to west, contrary to the order of the signs.

It is found from observation, that the equator and ecliptic do not always intersect each other in the same points, but that the points of intersection change their place, moving from east to west, whilst the inclination of the planes remains the same. This motion is called the *precession of the equinoxes*, because it carries the equinoctial points *in precedentia signa*.

## P R O P. CLXX.

The precession of the equinoxes makes the tropical year shorter than the periodical year.

If, while the sun moves in the order of the signs, the equinoctial point moves in the contrary direction, it is manifest, that the sun must arrive at the solstitial or equinoctial point from which it set out, before it arrives at the same place in the zodiac, or must complete the tropical year sooner than the periodical year.

The tropical year is observed to be 365 days 5 hours 49 minutes; the periodical year, 365 days, 6 hours, 4 minutes, 56 seconds.

P R O P.



## P R O P. CLXXI.

The precession of the equinoxes causes the poles of the equator to describe a circle from east to west about the poles of the ecliptic.

In this precession, the plane of the equator revolves from east to west, cutting the ecliptic, which, with its axis, is at rest, in successive points. But while the plane of the equator is revolving, its axis must revolve with it the same way. And, since the plane of the equator is always equally inclined to that of the ecliptic, the axis of the equator must always have the same inclination to the axis of the ecliptic: consequently, the poles of the equator will revolve round the poles of the ecliptic, always preserving the same distance from each other; that is, the poles of the equator will describe a circle about the poles of the ecliptic.

EXP. The precession of the equinoxes, and the revolution of the pole of the equator about that of the ecliptic, may be thus represented on the celestial globe. Let the broad wooden horizon represent the ecliptic; place the axis of the globe perpendicular to the wooden circle; the ecliptic on the globe will then make an angle of  $23^{\circ}.30'$  with the wooden horizon: consequently, if the wooden horizon represents the ecliptic, the circle which commonly represents the ecliptic will now represent the equator; and the two points in which this circle cuts the wooden horizon will represent the equinoctial points. If the globe, in this position, be turned slowly round from east to west, these points of intersection will move round the same way, while the inclination of the circle which now represents the equator to that which represents the ecliptic remains the same: whence the precession of the equinoxes is properly represented. Again, the axis and poles of the globe now representing those of the ecliptic, the axis and poles of the ecliptic, marked on the globe, will represent those of the equator; and in turning the globe round from east to west, the points which represent the poles of the equator, will revolve the same way round the poles of the globe which represent those of the ecliptic, and the axis of the supposed equator will always make the same angle with the plane of the supposed ecliptic.

## P R O P. CLXXII.

The precession of the equinoxes is caused by the action of the sun and moon on that excess of matter about the equatorial parts of the earth, by which from a perfect sphere it becomes an oblate spheroid.

Let ADCB be the plane of the ecliptic, S the sun, E the earth, and AFBG a ring encompassing the earth at any distance, as Saturn is encompassed by its ring. Let the

Plate II.  
Fig. 12.

T t 2

half



half of this ring AGB towards the sun be above the plane of the ecliptic, and the other half below it: then, a line passing through A and B will be the line of the nodes of this ring. If it be supposed that this ring moves round its center E, the same way in which the moon moves round the earth, it is obvious that every point of this ring will be acted upon by the disturbing force of the sun in the same manner as the moon was shewn to be acted upon in Prop. CLVIII. &c. Particularly, the motion of the nodes of this ring, and consequently of the whole ring which moves with these nodes, and its inclination to the plane in which its center moves, will be affected in the same manner with the orbit of the moon: whence, its nodes when in syzygies will stand still, and its inclination will be greatest; but in all other situations, the nodes will go backwards, and fastest of all when in the quadratures, at which time the inclination of the ring will be the least. This will be the case whatever be the thickness of the ring, or its distance from the center.

If this ring be supposed to adhere to the earth, it is obvious that it will still have the motions described above, and that in this situation, the earth itself must participate of these motions. Now the earth being an oblate spheroid, having its equatorial diameter longer than that which passes through its poles, this redundancy of matter, by which the form of the earth departs from a perfect sphere, may be considered as a portion of the supposed ring, which receives from the action of the sun the motions above mentioned, and communicates them to the earth. Hence the equinoctial points, which are the nodes of the ring, when they are in syzygy, that is, at the equinox, will stand still, and the inclination of the equator to the plane of the ecliptic will be the greatest: in all other situations they will go backwards, and fastest when in quadrature at the solstices; and the inclination of the plane of the equator to that of the ecliptic is then the least.

COR. Hence the axis of the earth, being perpendicular to the plane of the equator, changes therewith its inclination to the plane of the ecliptic twice in every revolution of the earth about the sun. For instance, it increases whilst the earth is moving from the solstitial to the equinoctial, and diminishes as much in its passage from the equinoctial to the solstitial points: which phenomenon is called the Nutation of the Poles.

SCHOL. This precession of the equinoxes is found to be 50 seconds of a degree, every year, westward or contrary to the sun's annual motion: so that with respect to the fixed stars, the equinoctial points fall backwards 30 degrees in 2160 years, whence the stars will appear to have gone 30 degrees forward, with respect to the signs of the ecliptic, which are reckoned from the equinoctial point. Thus the stars which were formerly in Aries are now in Taurus, &c. This period is completed in 25,920 years.



## C H A P. V.

## OF THE TIDES.

## P R O P. CLXXIII.

The tides are caused by the attraction of the moon and of the sun.

Let  $A\phi Ln$  be the earth, and  $C$  its center; let the dotted circle  $PN$  represent a mass of water covering the surface of the earth; let  $M, m$ , be the moon;  $S, s$ , the sun in different situations. Because the power of gravity diminishes as the squares of the distances increase, (by Prop. CXLVIII.) the waters on the side of the earth  $A$  are more attracted by the moon at  $M$  than the central parts of the earth  $C$ , and the central parts are more attracted than the waters on the opposite side of the earth at  $L$ : consequently (as was shewn concerning the moon) the waters on the side  $L$  will be as it were attracted *from* the center of the earth, or will recede from thence. Therefore, while the moon is at  $M$ , the waters will rise towards  $a$  and  $l$  on the opposite sides of the earth  $A, L$ ; while, by the oblique attraction of the moon, the waters at  $P$  and  $N$  will be depressed.

Plate II.  
Fig. 14.

Or thus: because (by Prop. CLV.) the moon and earth are continually revolving about their common center of gravity, suppose  $a$ ; the points  $A, C, L$ , describing circles about this common center in the same periodical times, the forces required to retain them in these circles (as may be inferred from Book II. Prop. LXXV.) will be to each other as their distances from the center  $aA, aC, aL$ . Consequently, the point  $L$  requires a greater force than  $C$ , and  $C$  than  $A$ , to retain it in its orbit. Now these points are retained in their respective circles by the moon at  $M$ ; and consequently the point  $L$ , which is most remote, and therefore requires the greatest force, is attracted the least, whilst  $A$ , the nearest point, is attracted the most. Thus, the water about  $A$  being attracted too much, and that about  $L$  too little, both will have their gravity diminished by the action of the moon, and will endeavour to leave the center  $C$ ; while the water at  $P$  and  $C$ , having their gravity increased by the same cause, will subside. Hence the form of the water on the surface of the earth will become an oblong spheroid.

This oval of waters keeps pace with the moon in its monthly course round the earth; while the earth, by its daily rotation about its axis, presents each part of its surface to the direct action of the moon, twice each day, and thus produces two floods and two ebbs. But because the moon is in the mean time passing from east to west in its orbit, it comes

to.



to the meridian of any place later than it did the preceding day; whence the two floods and ebbs require nearly 25 hours to complete them. The tide is at the greatest height, not when the moon is in the meridian, but some time afterwards, because the force by which the moon raises the tide continues to act for some time after it has passed the meridian.

As the moon thus raises the water in one place, and depresses it in another, the sun does the same; but in a much less degree, on account of the small ratio of the semidiameter of the earth to the distance of the sun; for, as it was shewn of the moon, that the force of the sun by which it disturbs its motion is as the distance of the moon from the earth to that of the sun from the same, so, in this case, the force of the sun to disturb the waters is as the semidiameter of the earth to the distance of the sun, which ratio is very small.

#### P R O P. CLXXIV.

The tides are greatest at the new and full moons, and least at the first and last quadratures, and the highest tides are near the time of the equinoxes.

Plate II.  
Fig. 14.

When the moon is in conjunction or opposition with the sun, as  $M, m, S$ , the tides which each endeavours to raise are in the same place; whereas, when the moon is in the first or last quarter, the sun being in the meridian when the moon is in the horizon, as  $M, s$ , depresses the water where the moon raises it; whence the tides are then the least of all. On the full and new moons, which happen about the equinoxes, when the luminaries are both in the equator or near it, the tides are the greatest of all: for first, the two eminences of water are at the greatest distance from the poles, and hence the difference between ebb and flood is more sensible; for if those eminences were at the poles, it is obvious we should not perceive any tide at all: secondly, the equatorial diameter of the earth produced passes through the moon, which diameter is longer than any other, and consequently there is greater disproportion between the distances of the zenith, center and nadir, from the center of gravity of the earth and moon, in this situation than in any other: and thirdly, the water rising higher in the open seas, rushes to the shores with greater force, where being stopped, it rises higher still; for it not only rises at the shores in proportion to the height it rises to in the open seas, but also according to the velocity with which it flows from thence against the shore. The spring tides, which happen a little before the vernal and after the autumnal equinox, are the greatest of all, because the sun is nearer the earth in the winter than in the summer.

#### P R O P. CLXXV.

When the moon is in the northern hemisphere, it produces a greater tide while it is in the meridian above the horizon, than when  
it



it is in the meridian below it; when in the southern hemisphere, the reverse.

Let AFD represent the earth, whose center is T, and axis PO, the point P the north pole, and O the south pole, EQ the equator, FH a parallel to it on the south side, and KD another parallel to it on the north side. Let the fluid surrounding the earth form itself into an oblong spheroid, whose longer axis HK produced, passes through the moon at L. The right lines TK, or TH, drawn from the center T, will represent the greatest height of the water in those places. Then, supposing NM perpendicular to KH, TN, or TM, will denote the least height, and will represent the height of the water in all parts of the globe through which the circle NM passes. The right lines TE, TF, TH, TQ, TD, will shew the height of the water in those respective places E, F, H, Q, D.

Plate 11.  
Fig. 15.

Let us now consider some place in particular, which, by the diurnal motion of the earth, describes the parallel KD. When that place is at K, the height of the water TK is the greatest, that is, it will be high water, and the moon L will be in the meridian. But afterwards, when that place comes to X, the height of the water will be the least, that is, it will be low water; and when the same place comes to D, it will be high water again. But because TK is greater than TD, therefore, in the present case, when the moon is on the north side the equator, or in the northern signs, the height of the sea, or tide, will be greater when the moon is in the part of the meridian which is above the horizon, than when it is in the meridian, and below it. Hence it is that the moon, when it is in the northern signs, makes the greatest tides on our side the equator when it is above the earth.

Again, TH, on the south side the equator, is longer than TF; and therefore, to a place that describes the parallel FH, the greatest height of the water, when the moon is in the northern signs, is when it is on that part of the meridian that is below the horizon of that place, and the least tides when it is above the horizon. For the like reason, when the moon is in the southern signs, the greatest tides on the other side of the equator will be when it is below our horizon, and the least tides when it is above it.

SCHOL. What hath been said of the tides, must be understood upon supposition, that the globe of the earth is covered entirely with water to a considerable depth: but continents which stop the tide, freights between them, islands, and the shallowness of the sea in some places, which are all impediments to the course of the water, cause many exceptions to what hath been above laid down. These exceptions can only be explained from particular observations on the nature of tides at different places.



B O O K V. P A R T III.

O F T H E F I X E D S T A R S.

DEF. LXVII. Those bodies which always appear in the heavens at the same distance from each other, are called *Fixed Stars*.

P R O P. CLXXVI.

The fixed stars are luminous bodies.

Because they appear as points of small magnitude when viewed through a telescope, they must be at such immense distances, as to be invisible to the naked eye if they borrowed their light; as is the case with respect to the satellites of Jupiter and Saturn, although they appear of very distinguishable magnitude through a telescope.

P R O P. CLXXVII.

The fixed stars appear of different magnitudes.

The difference in the apparent magnitude of the stars is such as to admit of their being divided in six classes, the largest being called stars of the first magnitude, and the least, which are visible to the naked eye, stars of the sixth magnitude. Stars only visible by the help of glasses are called *telescopic stars*.

P R O P. CLXXVIII.

The fixed stars are divided into constellations, or systems of stars.

The ancients, that they might the better distinguish the stars with regard to their situation in the heavens, divided them into several constellations, that is, systems of stars,



stars, each system consisting of such as are near each other. And to distinguish these systems from one another, they gave them the names of such men or things as they fancied the space they took up in the heavens represented. To these several new constellations have been added by modern astronomers.

SCHOL. The following table contains the names of the constellations, and the number of stars observed in each by different astronomers.

| The antient Constellations.    |                    | Ptolemy. | Tycho. | Hevelius. | Flamsteed. |
|--------------------------------|--------------------|----------|--------|-----------|------------|
| Urfa minor                     | The Little Bear    | 8        | 7      | 12        | 24         |
| Urfa major                     | The Great Bear     | 35       | 29     | 73        | 87         |
| Draco                          | The Dragon         | 31       | 32     | 40        | 80         |
| Cepheus                        | Cepheus            | 13       | 4      | 51        | 35         |
| Bootes, <i>Arctophilax</i>     | Bootes             | 23       | 18     | 52        | 54         |
| Corona Borealis                | The Northern Crown | 8        | 8      | 8         | 21         |
| Hercules, <i>Engonasin</i>     | Hercules kneeling  | 29       | 28     | 45        | 113        |
| Lyra                           | The Harp           | 10       | 11     | 17        | 21         |
| Cygnus, <i>Gallina</i>         | The Swan           | 19       | 18     | 47        | 81         |
| Cassiopea                      | Cassiopea          | 13       | 26     | 37        | 55         |
| Perseus                        | Perseus            | 29       | 29     | 46        | 59         |
| Auriga                         | The Waggoner       | 14       | 9      | 40        | 66         |
| Serpentarius, <i>Ophiuchus</i> | Serpentarius       | 29       | 15     | 40        | 74         |
| Serpens                        | The Serpent        | 18       | 13     | 22        | 64         |
| Sagitta                        | The Arrow          | 5        | 5      | 5         | 18         |
| Aquila, <i>Vultur</i>          | The Eagle          | 15       | 12     | 23        | 71         |
| Antinous                       | Antinous           |          | 3      | 19        |            |
| Delphinus                      | The Dolphin        | 10       | 10     | 14        | 18         |
| Equulus, <i>Equi sectio</i>    | The Horse's Head   | 4        | 4      | 6         | 10         |
| Pegasus, <i>Equus</i>          | The Flying Horse   | 20       | 19     | 38        | 89         |
| Andromeda                      | Andromeda          | 23       | 23     | 47        | 66         |
| Triangulum                     | The Triangle       | 4        | 4      | 12        | 16         |
| Aries                          | The Ram            | 18       | 21     | 27        | 66         |
| Taurus                         | The Bull           | 44       | 43     | 51        | 141        |
| Gemini                         | The Twins          | 25       | 25     | 38        | 85         |
| Cancer                         | The Crab           | 23       | 15     | 29        | 83         |
| Leo                            | The Lion           | 35       | 30     | 49        | 95         |
| Coma Berenices                 | Berenice's Hair    |          | 14     | 21        | 43         |
| Virgo                          | The Virgin         | 32       | 33     | 50        | 110        |
| Libra, <i>Chelæ</i>            | The Scales         | 17       | 10     | 20        | 51         |
| Scorpius                       | The Scorpion       | 24       | 10     | 20        | 44         |
| U u                            |                    | The      |        |           |            |



| The antient Constellations. |                            | Ptolemy. | Tycho. | Hevelius. | Flamsteed. |
|-----------------------------|----------------------------|----------|--------|-----------|------------|
| Sagittarius                 | The Archer                 | 31       | 14     | 22        | 69         |
| Capricornus                 | Capricorn                  | 28       | 28     | 29        | 51         |
| Aquarius                    | The Water-bearer           | 45       | 41     | 47        | 108        |
| Pisces                      | The Fishes                 | 38       | 36     | 39        | 113        |
| Cetus                       | The Whale                  | 22       | 21     | 45        | 97         |
| Orion                       | Orion                      | 38       | 42     | 62        | 78         |
| Eridanus, <i>Fluvius</i>    | Eridanus, <i>the River</i> | 34       | 10     | 27        | 84         |
| Lepus                       | The Hare                   | 12       | 13     | 16        | 19         |
| Canis major                 | The Great Dog              | 29       | 13     | 21        | 31         |
| Canis minor                 | The Little Dog             | 2        | 2      | 13        | 14         |
| Argo Navis                  | The Ship                   | 45       | 3      | 4         | 64         |
| Hydra                       | The Hydra                  | 27       | 19     | 31        | 60         |
| Crater                      | The Cup                    | 7        | 3      | 10        | 31         |
| Corvus                      | The Crow                   | 7        | 4      |           | 9          |
| Centaurus                   | The Centaur                | 37       |        |           | 35         |
| Lupus                       | The Wolf                   | 19       |        |           | 24         |
| Ara                         | The Altar                  | 7        |        |           | 9          |
| Corona Australis            | The Southern Crown         | 13       |        |           | 12         |
| Pisces Australis            | The Southern Fish          | 18       |        |           | 24         |

## The New Southern Constellations.

|                              |                      |    |
|------------------------------|----------------------|----|
| Columba Noachi               | Noah's Dove          | 10 |
| Robur Carolinum              | The Royal Oak        | 12 |
| Grus                         | The Crane            | 13 |
| Phoenix                      | The Phenix           | 13 |
| Indus                        | The Indian           | 12 |
| Pavo                         | The Peacock          | 14 |
| Apus, <i>Avis Indica</i>     | The Bird of Paradise | 11 |
| Apis, <i>Musca</i>           | The Bee or Fly       | 4  |
| Chamæleon                    | The Cameleon         | 10 |
| Triangulum Australis         | The South Triangle   | 5  |
| Piscis volans, <i>Passer</i> | The Flying Fish      | 8  |
| Dorado, <i>Xiphias</i>       | The Sword Fish       | 6  |
| Toucan                       | The American Goose   | 9  |
| Hydrus                       | The Water Snake      | 10 |

Hevelius's



*Hevelius's Constellations made out of the unformed Stars.*

|                   |                   | <i>Hevelius.</i> | <i>Flamsteed.</i> |
|-------------------|-------------------|------------------|-------------------|
| Lynx              | The Lynx          | 19               | 44                |
| Leo minor         | The Little Lion   |                  | 53                |
| Afterion & Chara  | The Greyhounds    | 23               | 25                |
| Cerberus          | Cerberus          | 4                |                   |
| Vulpecula & Anser | The Fox and Goose | 27               | 35                |
| Scutum Sobieski   | Sobieski's Shield | 7                |                   |
| Lacerta           | The Lizard        | 10               | 16                |
| Camelopardalus    | The Camelopard    | 32               | 58                |
| Monoceros         | The Unicorn       | 19               | 31                |
| Sextans           | The Sextant       | 11               | 41                |

SCHOL. 1. Stars not included in any constellation are called *unformed* stars. Besides the names of the constellations, the antient Greeks gave particular names to some single stars, or small collections of stars: thus, the cluster of small stars in the neck of the bull, was called the *Pleiades*; five stars in the bull's face, the *Hyades*; a bright star in the breast of *Leo*, the *Lyon's Heart*; and a large star between the knees of *Bootes*, *Arcturus*.

SCHOL. 2. The constellations may be represented on two plane spheres projected on a great circle, or on the convex surface of a solid sphere, as on the celestial globe, or most perfectly on the concave surface of a hollow sphere. If the celestial globe be made use of, after rectifying it to the time of the night, the stars may be found, by conceiving a line drawn from the center of the globe through any star in the heavens, and its representation upon the globe. Greek letters have been added by Bayer to stars in the several constellations of his catalogue ( $\alpha$  being affixed to the largest star) by means of which any star may be easily found.

## P R O P. CLXXIX.

That luminous part of the heavens, called the *Milky Way*, consists of fixed stars too small to be seen by the naked eye.

This is found from observations made with telescopes.

SCHOL. 1. There are spots in the heavens, called *Nebulæ*, some of which consist of clusters of telescopic stars, others appear as luminous spots of different forms. The most considerable is one in the mid-way between the two stars on the blade of Orion's sword, marked  $\theta$  by Bayer.

SCHOL. 2. New stars sometimes appear, while others disappear. Several stars, mentioned by antient astronomers, are not now to be found: several are now visible to



the naked eye, which are not mentioned in the antient catalogues; and some stars have suddenly appeared, and again, after a considerable interval, vanished: also, a change of place has been observed in some stars.

SCHOL. 3. The number of the stars is unknown. The catalogue published by Bayer contains 1160; that by Flamsteed, which includes many telescopic stars, contains 3000; and farther improvements in telescopes are continually enabling astronomers to enlarge the catalogue.

## P R O P. CLXXX.

The longitude of the fixed stars increaseth, while their latitude remains the same.

Because the vernal equinoctial point (by Prop. CLXXI.) moves westward, the distance between any given star and that point, that is, its longitude, will increase. But since this change is produced by the precession of the equinoxes, which is performed round the axis of the ecliptic, this motion will make no change in the distance of the fixed star from the ecliptic, that is, in its latitude.

COR. Hence the constellations of the zodiac are to the east of those signs or arcs of the zodiac which are called by the same names. The first part of the constellation Aries, by the precession of the equinoxes, has gone so far to the east, since the names were first given to the signs, that it is now  $30^{\circ}$ . from the first degree of Aries in the line of the ecliptic.

DEF. LXVIII. The *Annual Parallax* of a heavenly body, is the change of its apparent place, as it is viewed from the earth in its annual motion.

Plate II.  
Fig. 16.

If ADBC be the orbit of the earth, S the sun, and A, B, the earth in opposite parts of its orbit; the change in the apparent place of any body, as viewed from A and from B, is its annual parallax.

## P R O P. CLXXXI.

The annual parallax of any heavenly body is proportional to the angle which a diameter of the earth's orbit would subtend, if it was viewed from that body.

If



If when the earth is at A, the fixed star E appears at or near the pole, and when the earth has passed to the opposite point B, a different star F appears at or near the same pole, the star E will have changed its place in respect of the pole; for when the pole is at F, the star E, which was at or near it before, is at the distance EF from it: the apparent length of this distance EF (by Def. LXVIII.) is the star's annual parallax. Now if AB, a semidiameter of the earth's orbit, was to be viewed from the star E, it would subtend the angle AEB; but, because the axis of the earth is always parallel to itself, AE and BF, which coincide with the axis, are likewise parallel; whence (El. I. 29.) the angle EBF, subtended by EF, is equal to AEB, subtended by AB; and AEB is the parallactic angle.

COR. The annual parallax of any heavenly body is inversely as its distance from the earth: for the angle AEB (by Book IV. Prop. LXIX.) is inversely as the distance of AB, the axis of the earth's orbit.

## P R O P. CLXXXII.

The fixed stars have no sensible annual parallax.

When the place of the star E is observed by the best instruments from opposite points of the earth's orbit, its apparent place in the heavens remains the same, which could not be the case if the angle of its parallax were so much as one second.

COR. Hence it appears, that the fixed stars are so remote that a diameter of the earth's orbit bears no proportion to their distance, or (by Prop. CLXXXI.) that a diameter of the earth's orbit, if viewed from one of the fixed stars, would appear as a point.

## P R O P. CLXXXIII.

The motion of the earth, and the progressive motion of light, will make a fixed star, which has no sensible parallax, deviate from its true place in the direction in which the earth moves.

If a star S passes through the zenith of any place when the earth is at A, it will (by last Prop.) pass through the zenith of the same place when the earth is at B, the opposite extremity of the earth's orbit. Consequently, such a star might be seen through a vertical telescope in the same perpendicular at any point of the earth's orbit, if the motion of light from the star were instantaneous. But the progressive motion of light will cause the star to deviate from the perpendicular; for, let the earth be moving from B to A, and let the velocity of light be to the velocity of the earth, as CA to BA, and let CB be the diagonal of the parallelogram formed from CA, BA. Then the direction of

Plate II.  
Fig. 17.



of a telescope, in order to see the star  $S$  when the earth is arrived at  $A$ , must be  $AH$ , parallel to  $BC$ . For, suppose  $BC$  to be a very long slender telescope, through which only one ray of light could pass at a time, or to be the axis of a larger telescope. The star  $S$  cannot be seen through this telescope, but through a telescope perpendicular to  $B$ , if the earth be stationary at  $B$ , and the progress of light instantaneous. But if the telescope in the position  $BC$  were to continue in this position, and to move along with the earth to  $A$ , so as to come into the situation  $AH$ , when the earth arrives at  $A$ , the star  $S$  might then be seen through it. For, since the straight course of the ray is the line  $CA$ , in which it must always be if it comes to the eye without interruption; and since the ray cannot come directly along  $CB$  the axis of the telescope, and arrive at the eye in this axis, unless it is always in the axis; that is, since the ray, in order to come to the eye must be always in the line  $CA$ , and also in the line  $CB$ , it must be always in the common intersection of these two lines. Now  $C$  is the common intersection when the earth is at  $B$ ;  $e$  is the common intersection when the earth is at  $E$ ;  $f$ , when it is at  $F$ ;  $g$ , when it is at  $G$ ; and  $A$ , when at  $A$ ; the telescope, at each station, being successively in the situations  $CB$ ,  $EE$ ,  $FF$ ,  $GG$ ,  $HA$ . Thus the common intersection descends down the line  $CA$ , while the earth moves from  $B$  to  $A$ ; and, since the velocity of light is to that of the earth, as  $CA$  to  $BA$ , a ray of light will likewise have descended down  $CA$ , while the earth was moving from  $B$  to  $A$ . Therefore, in the whole motion of the telescope, the ray will have been in the common intersection of the line  $CA$ , and the axis of the telescope, and consequently will have passed along the axis of the telescope, and will come without interruption to the eye at  $A$ .

Thus it appears, that by the progressive motion of light, a ray which, coming from  $S$ , enters, at  $C$ , a telescope in the situation  $CB$ , will arrive at the eye, when the telescope, carried along  $BA$  with a velocity which is to that of the ray of light, as  $BA$  to  $CA$ , is come into the situation  $HA$ ; and consequently (Book IV. Prop. II.) the eye will see the star through the telescope in the direction  $AH$  the axis of the telescope; that is, some point in the line  $AH$  produced will be the apparent place of the star. Thus the star's apparent place has deviated from its true place  $S$  in the direction  $BA$ , in which the earth was moving, so that if the motion of the earth is from north to south, the star which appeared in the zenith of the place when the earth was at  $B$ , will appear to the southward of the zenith when the earth is arrived at  $A$ , and the reverse when the earth is moving from south to north.

According to Bradley's observations, made on the star  $\gamma$  in the constellation Dragon, this star deviated southward from the zenith from December till March, when it had departed from the zenith  $20''$ . From that time till June its southern deviation decreased, after which it deviated northward, and in September appeared about  $20''$ . towards the north of its station in June, from which time till December, it continued returning to its first situation. Thus the deviation of the star was always in the direction of the earth's motion, and contrary to that of any deviation which might be



be supposed to arise from the annual parallax of the star. But such a deviation could not happen unless the earth moved, and the motion of light was progressive; for if the earth did not move, since the star is fixed, no alteration could be made in the apparent place of the star by the progressive motion of its rays in a vertical direction: and if the earth moves, and the propagation of light were instantaneous, the earth's velocity would be nothing in respect of the velocity of light, or BA with respect to CA would be nothing; whence the angle ACB, and its alternate angle CAH, would vanish, and AH would become coincident with AC, and consequently, the star would have no deviation from its true place. Hence we may conclude from the deviation of the star above described, both that the earth moves, and that the motion of light is progressive.

SCHOL. From these observations it is found, that the velocity of star-light is such as carries it through a space equal to the sun's distance from the earth in  $8'. 13''$ .







# B O O K VI.

## O F M A G N E T I S M.

### PROPOSITION I.

That mineral substance which is called the Loadstone, or Magnet, has the property of attracting iron, and no other body whatever unless it has a mixture of iron.

EXP. I. Shew the action of the magnet on needles, and on steel filings strewed lightly over any smooth surface.

2. Let a needle be suspended from a loadstone, and a string passing through its eye be fastened to the beam of a balance placed under it; the degree of force with which it is attracted may be measured.

### P R O P. II.

The magnetic power may be communicated from the loadstone to iron, by the touch or gentle friction, and from one piece of iron to another, which then becomes an artificial magnet; and this communication of power is without apparent loss of power in the loadstone.

EXP. Let the power be communicated to a needle from the loadstone.

### P R O P. III.

The action and re-action of the magnetic power are mutual and equal.

EXP. Two equal needles, suspended freely, will be attracted with equal force.



## P R O P. IV.

If a magnet has free motion, one of its extremities will be directed towards the north, the other towards the south; these are called its poles.

EXP. In the mariner's compass the needle points nearly north and south.

## P R O P. V.

If the north pole of a magnet be drawn along from the middle of a needle towards either extremity, that part of the needle which is touched by the magnet will be directed to the south: if the south pole of a magnet be applied in the same manner, the part which is touched will be directed to the north.

EXP. 1. Touch two needles according to the Proposition, and observe their direction when suspended freely.

2. Exhibit the method of making artificial magnets.

## P R O P. VI.

There is a small variation in the direction of the magnetic needle, called its declination, which differs in degree at different places and times.

This is known by observing the different points of the compass at which the sun rises or sets, and comparing them with the true points of the sun's rising or setting, according to astronomical tables.

SCHOL. The following table shews the mean declination of the needle at different times in Paris and London.

| Year. | Paris.       | Year. | London.      |
|-------|--------------|-------|--------------|
| 1580  | 11°. 30'. E. | 1576  | 11°. 15'. E. |
| 1610  | 8°. E.       | 1634  | 4°. 5'. E.   |
| 1640  | 3°. E.       | 1657  | 0°. 0'.      |
| 1666  | 0°. 0'.      | 1665  | 1°. 22'. W.  |
| 1670  | 1°. 30'. W.  | 1692  | 6°. W.       |
| 1700  | 8°. 12'. W.  | 1730  | 16°. 15'. W. |
| 1728  | 14°. 0'. W.  | 1756  | 15°. 15'. W. |
| 1771  | 19°. 45'. W. | 1774  | 21°. 16'. W. |
|       |              | 1776  | 21°. 47'. W. |

Near



Near the equator, in long.  $40^{\circ}$ . East, the highest variation from the year 1700 to 1756, was  $17^{\circ}.15'$ . West; and the least  $16^{\circ}.30'$ . W. In lat.  $15^{\circ}$ . N. and long.  $60^{\circ}$ . W. the variation was constantly  $5^{\circ}$ . E. In lat.  $10^{\circ}$ . South, and long.  $60^{\circ}$ . E. the variation decreased from  $17^{\circ}$ . W. to  $7^{\circ}.15'$ . W. In lat.  $10^{\circ}$ . S. and long.  $5^{\circ}$ . W. it increased from  $2^{\circ}.15'$ . to  $12^{\circ}.45'$ . W. In lat.  $15^{\circ}$ . N. and long.  $20^{\circ}$ . W. it increased from  $1^{\circ}$ . W. to  $9^{\circ}$ . W. In the Indian seas the irregularities were greater, for in 1700, the west variations seem to have decreased regularly from long.  $50^{\circ}$ . E. to long.  $100^{\circ}$ . E. but in 1756, the variation decreased so fast, that there was east variation in long.  $80^{\circ}$ .  $85^{\circ}$ . and  $90^{\circ}$ . E. and yet, in long.  $95^{\circ}$ . and  $100^{\circ}$ . E. there was west variation.

In the year 1775, in lat.  $58^{\circ}.17'$ . S. and long.  $348^{\circ}.16'$ . E. it was  $0^{\circ}.16'$ . W. In lat  $2^{\circ}.24'$ . N. and long.  $32^{\circ}.12'$ . W. it was  $0^{\circ}.14'.45''$ . W. In lat.  $50^{\circ}.6'.30''$ . N. and long.  $4^{\circ}.0'$ . W. it was  $19^{\circ}.28'$ . W.

### P R O P. VII.

A needle which, before it receives the magnetic power, rests on its center parallel to the horizon, on becoming magnetical will incline towards the earth: this is called its inclination, or dipping.

EXP. A needle turning in equilibrio on a pointed wire, after being touched by the magnet, will dip.

SCHOL. The inclination or dip of the magnetic needle at London in 1576, was  $71^{\circ}.50'$ . In 1653,  $72^{\circ}.45'$ . In 1676,  $73^{\circ}.30'$ . In 1720,  $75^{\circ}.10'$ . In 1723, about  $75^{\circ}$ . &c. The mean dip of the needle at London in 1775 and 1776, was  $72^{\circ}.30'$ .

A daily variation is observed both in the declination and inclination of the needle.

### P R O P. VIII.

Two magnets, having a free motion, will attract when different poles are directed towards each other, and repel when the adjacent poles are of the same kind.

EXP. A needle turning on its center will be attracted or repelled by another, as different or the same poles are brought near to each other.

SCHOL. If the magnetic powers are very unequal, or the two bodies are forcibly brought together, they will attract with the same poles.

EXP. I. Suspend a magnet by a thread, and let a small needle be brought near it, at the same poles.

2. Bring two <sup>very unequal</sup> needles into contact at the same poles, suspended in the same manner, they will cohere.



## P R O P. IX.

The action of the magnetic power is not retarded or increased by the interposition of any body whatsoever which is not magnetic, or capable of magnetism.

Exp. 1. Move steel filings, placed on a brass plate, in water, &c. by holding a magnet under the vessel.

2. Sprinkle steel dust on a sheet of paper, under which is placed a magnet, or two magnets having their poles opposite to each other and at the distance of about an inch.

3. A needle under an exhausted receiver will be attracted at the same distance as in the open air.

## P R O P. X.

If a magnet be cut through the middle, each piece will become a complete magnet, and the parts which were contiguous will become opposite poles.

Exp. Cut a magnetic needle into two parts, and present the north pole of a magnet to each end of the parts.

## P R O P. XI.

A bar of iron acquires a magnetic power, by remaining long in a situation perpendicular to the surface of the earth, or by being hammered in the magnetic line. The power of a magnet will be diminished if it be not kept in the magnetic line; if it communicates virtue to a piece of iron much larger than itself; or if a piece of iron be not kept in contact with, or near to one of its poles. This power may be wholly destroyed by fire.

This Proposition is confirmed by experience.



# B O O K VII.

## O F E L E C T R I C I T Y.

### PROPOSITION I.

There is in nature a substance, called the **ELECTRIC FLUID**, which, being excited, becomes perceptible to the senses.

EXP. 1. Let a long glass tube be rubbed with the hand, or with a leathern cushion; the electric fluid, being thus excited, will attract light substances, and give a lucid spark to the finger, or any metallic substance, brought near it.

2. Let a large glass cylinder, turned on an axis by means of a wheel connected with it by a cord, and rubbing against a cushion, be put into motion; this instrument, called an **Electrical Machine**, increasing the excitation, will exhibit the phenomena of the electric fluid more perfectly than a glass tube.

### P R O P. II.

The electric fluid passes easily along the surfaces of some bodies; whilst other bodies do not convey it: the former are called *Conductors*, the latter, *Non-conductors*, or *Electrics*.

EXP. A metallic cylinder, being fixed upon glass supporters, and placed near the electric machine, will, by means of pointed wires, receive the electric fluid from the glass cylinder, and the fluid will be diffused over the whole surface of the metallic cylinder, from whence it cannot pass through the glass supporters which are electric, but may be conveyed away by any metallic or other conducting substances, brought near, or into contact with it. This metallic cylinder is called the *Prime Conductor*, or the *Conductor*.



## P R O P. III.

Some conductors are more perfect than others; and the electric fluid passes through that which is most perfect.

Exp. The fluid will pass through a wire held in the hand.

SCHOL. The following bodies are conductors and electrics, disposed in the order of their degrees of perfection: CONDUCTORS; gold, silver, copper, brass, iron, tin, quicksilver, lead, the semi-metals, ores, charcoals, water, ice, snow, salts, soft stones, smoke, steam: NON-CONDUCTORS, or ELECTRICS; glass, precious stones, resins, gums, amber, sulphur, baked wood, bituminous substances, wax, silk, cotton, feathers, wool, hair, paper, air, oil, hard stones. Many electrics become conductors when heated, and all when moistened.

## P R O P. IV.

Non-conductors retain the fluid on a small part of their surface where the friction has acted; conductors diffuse it over all their surface, and therefore cannot confine it, unless they be surrounded entirely by non-conductors, or be *insulated*.

Exp. Observe the partial distribution of the fluid on the excited glass cylinder, and its universal diffusion over the metallic conductor.

## P R O P. V.

The electric fluid may be excited by rubbing, by pouring a melted electric into another substance, and by heating and cooling.

Exp. 1. In working the electrical machine, the fluid is excited by friction.

2. Let sulphur be melted in an earthen vessel, and left to cool upon a conductor; when cold, it will be electrified, and attract light bodies.

3. The tourmalin stone will be electrified by increasing or diminishing its heat.

## P R O P. VI.

The electric fluid may be lodged in electrics, or in insulated conductors, in a greater quantity than naturally belongs to them, or they may be *positively* electrified.

Exp.



EXP. In working the machine, the cylinder acquires more than its natural quantity of fluid by excitation, the conductor, by communication: for, while there is a free conveyance of fluid from the earth to the rubber, by means of a conducting supporter, the conductor will be highly electrified; but if an electric supporter be placed under the rubber, the conductor will be electrified only in a small degree.

## P R O P. VII.

The electric fluid being accumulated on any body, will pass to any conductor brought near to the body: if it pass from, or be received by, pointed wires, it will be conveyed in a continued stream; if it pass from, or be received by, a surface which has no sharp points, it will be discharged with an instantaneous explosion or spark.

EXP. 1. Receive the fluid from the conductor upon a pointed wire, and upon a brass ball.

2. The fluid will be diffused through the surrounding atmosphere, by wires placed upon the conductor.

COR. Hence arises the necessity of keeping the whole surface of the conductor free from points.

SCHOL. When a conductor is electrified by communication, its whole electric power is discharged at once, on the near approach of a conductor communicating with the earth; whereas an excited electric, in the same circumstances, loses its electric power only in the parts near to the conductor.

## P R O P. VIII.

If conductors be insulated, they will retain a greater or less quantity of the electric fluid (the power of the machine being given) proportional to the extent of surface in the conductor.

EXP. Observe the difference in the magnitude and distance of sparks taken from a small conductor, and of those taken from a large one.

## P R O P. IX.

A body may be deprived of part of its natural portion of electric fluid, or be *negatively* electrified.

EXP.



Exp. If the rubber which communicates the fluid to the glass cylinder, and from thence to the conductor be insulated, because by working the machine a quantity of its fluid is conveyed away, and it cannot receive a fresh supply through its supporter, it will be in an exhausted or negative state.

## P R O P. X.

When bodies are negatively electrified they receive the fluid from other bodies brought near them.

Exp. 1. Let two insulated conductors, one of which is connected with the glass cylinder, the other with the rubber, be electrified; whilst they are in this state let them be brought near each other; a spark will pass from that which (by Prop. VI.) is positively, to that which (by Prop. IX.) is negatively, electrified.

2. Let two persons standing on glass feet be electrified, first both positively, or both negatively, they will not, on contact, communicate the fluid to each other; but let them be electrified, the one positively and the other negatively by making a communication from one to the conductor, and from the other to the rubber, on contact, the former will give, and the latter receive a spark.

## P R O P. XI.

From a pointed body positively electrified the fluid will be seen to stream out, towards any electrified body brought near it, in a conical pencil of rays; whereas in passing from the unelectrified body to a pointed body negatively electrified, it will form a globular flame, or star, about its point.

Exp. 1. Observe, in a dark room, the different appearances of the electric fluid at the extremity of a pointed wire, when the point is presented to an insulated conductor positively, and when it is presented to one negatively, electrified; or when such a wire is fixed upon a conductor positively or negatively electrified,

2. Within a *luminous conductor* electrified positively, (viewed in a dark room) the fluid will be seen passing in the form of a pencil from one wire, and received in the form of a star upon the other; and the reverse if it be electrified negatively.

P R O P.



## P R O P. XII.

If two bodies be electrified, both positively, or both negatively, they repel each other; but if one be electrified positively, and the other be negatively or not at all electrified, they attract each other.

EXP. 1. Light feathers, or hair, connected with the conductor, appear repellent, but are attracted by bringing any non-electrified body near them.

2. The hair of a person electrified becomes repellent.

3. In the graduated electrometer the ball is repelled according to the degree in which the conductor is electrified.

4. Downy feathers, paper figures, threads of flax, thistle down, gold leaf, brass dust, or other light bodies, brought near to the conductor, are alternately attracted and repelled. This will not take place if the bodies be laid on a plate of glass.

5. Two bells being suspended by wires from a brass rod connected with the conductor, and a third by a silk cord, and two small balls of brass suspended by a silken thread between the bells, the fluid will be communicated from the conductor to the outer bells, and by the balls to the middle bell, and from thence conveyed by a chain to the earth: the balls in receiving and communicating the fluid are attracted and repelled successively, and produce ringing.

6. Let water flow from a capillary tube, from which, before it is electrified, it passes in drops; upon being electrified, the particles of fluid will be separated, and their motion accelerated.

These appearances will be presented, whether the conductor be positively or negatively electrified.

COR. Since it is found that rubbed glass electrifies any insulated conductor positively, it may be determined whether any body is electrified positively or negatively, by bringing it near to a pith ball, or down-feather, positively electrified, and observing whether the ball or feather be attracted or repelled by the body.

EXP. Bring a pith-ball or down-feather, suspended by a silken thread and positively electrified by any rubbed glass surface, near to another pith-ball or feather suspended by a flaxen thread from a conductor connected with the cylinder; then bring the same near to a conductor connected with the rubber.

## P R O P. XIII.

The electric power will be conveyed through an insulated conductor of any length.

EXP. At the remote end of a long circuit of wire, suspended by silk cords, hang a brass plate: light bodies, brought near it, will be attracted.

P R O P.



## P R O P. XIV.

From the sharp points of electrified bodies there proceeds a current of air.

Exp. 1. A wire, with sharp points bended in opposite directions, and suspended on the point of a perpendicular wire inserted in the conductor, will be carried round by the current proceeding from the points.

2. Let several pieces of gilt paper be stuck like vanes into the side of a cork, through the center of which a needle passes; suspend the whole by a magnet, and present one of the vanes to the point of a wire inserted in the conductor; they will be put into motion.

## P R O P. XV.

Some bodies upon being rubbed, are electrified positively, and others negatively: and the same bodies are capable of being electrified positively, or negatively, as they are rubbed with different substances.

Exp. 1. Smooth glass becomes positively electrified by being rubbed with any substance hitherto tried, except the back of a living cat; rough glass becomes positively electrified by being rubbed with dry oiled silk, sulphur and metals; negatively, with woollen cloth, sealing-wax, paper, the human hand. White silk becomes positively electrified by being rubbed with black silk, metals, black cloth; negatively, with paper, hairs, the hand. Black silk will be positively electrified with red sealing-wax; negatively, with hare's skin, metals, the hand. Sealing-wax will be positively electrified with the hand, leather, woollen cloth, paper, hare's skin. Baked wood will be positively electrified with silk, negatively, with flannel. If these and other substances, being electrified, be brought near to a pith ball or down-feather, as described Prop. XII. Cor. Exp. it will appear whether they are electrified positively or negatively.

2. The tourmalin, while it is heating or cooling, is electrified positively on one side and negatively on the other; it is electrified only on two opposite sides, which may be called its poles, which lie in a right line with the center of the stone, and in the direction of its strata; and either side will be electrified positively or negatively, as it is applied to different substances. Other hard precious stones have been found to possess similar properties.



P R O P. XVI.

Bodies insulated, if placed within the influence of an electrified body, will be electrified, at the part adjacent to that body, in the manner contrary to that of the electrified body.

EXP. 1. Bring a conductor (without pointed wires) near to the glass cylinder, whilst the machine is working; if the conductor be not insulated, it will be negatively electrified till it is brought so near as to receive sparks from the cylinder: if the conductor be insulated, it will, in the same situation, be electrified negatively, in the parts nearest the cylinder, and positively in the parts more remote; as may be seen by bringing an excited glass tube (which is positively electrified) near to a ball suspended from the conductor. Compare Prop. XII. Cor.

2. Let two pith-balls be so suspended by flaxen threads as to be in contact when unelectrified; on being brought near to a body electrified positively, they will repel each other, being electrified negatively: if the balls be suspended in the same manner by silken threads, they will, in the same situation, be positively electrified.

3. Let a circular plate composed of rosin and sulphur, or of sealing wax, be negatively electrified by rubbing it with flannel; whilst it is in this state, let a metallic plate, of the same form and size, having a glass handle fastened to its center, be placed, by means of the handle, on the electrified plate; then receive a spark from the metallic plate with the finger: after which the metallic plate, being removed by the glass handle, will be found to be positively electrified. This instrument is called an electrophorus.

4. Let one side of a plate of glass be electrified positively, the other side will attract light bodies, being negatively electrified.

5. Let a plate of glass be placed between two metallic plates about two inches in diameter smaller than the plate of glass, and let the plates be supported by a conductor; upon positively electrifying the upper metallic plate, by means of a wire connected with the conductor, the fluid not being able to pass along the glass, will be accumulated upon the part contiguous to the upper metallic plate; whilst the lower metallic plate, being within the electric influence of the upper, will be negatively electrified.

P R O P. XVII.

When any electric substance is electrified, it will continue in that state till some conductor conveys away the accumulated or restores the deficient fluid; which will be done more or less rapidly, ac-

Z z

cording



according to the degree of conducting power in the conductor, and the number of points in which it touches the electric.

EXP. 1. When the metallic plate in the electrophorus is electrified (as described Prop. XVI. Exp. 3.) by setting it upon the electric plate, touching it with the finger, and separating it successively, many sparks may be obtained, without again exciting the electric plate; for this plate being negatively electrified, the metallic plate, on being touched with the hand, becomes positively electrified (by Prop. XVI.) and the electric plate remains long in its negative state, because, not being a conductor, its deficiency will be slowly supplied from the air where its surface is not covered.

2. If a glass vessel, held in the hand, receive the electric fluid on the inside from a wire fixed on the conductor, pith balls, placed under the vessel upon a conducting supporter, will continue long in motion.

3. Let a plate of glass be electrified in the manner described in Prop. XVI. Exp. 5. Because one side of the plate is positively electrified, and the other negatively, if a communication is made from one metallic plate to the other by means of some conductor, part of the accumulated fluid will suddenly pass to the side which is deficient; upon a second application of the plates of metal to the glass, there will be a second explosion.

#### P R O P. XVIII.

If a glass plane, or cylindrical vessel, *coated* on both sides with tinfoil or any other conducting substance, be *charged*, that is, positively electrified on one side, and consequently negatively electrified on the other; a communication being made from one side to the other by some conductor, the plane, or vessel, will be suddenly *discharged*, with an explosion.

There is a strong attraction (compare Prop. XII. and XVI.) between the fluids on opposite sides of the glass, or the fluid which is accumulated on one side makes a powerful effort towards the other side where the fluid is deficient; but, the substance of the glass itself being impervious to the electric fluid, the accumulated fluid cannot pass to the deficient side till a communication is made between them by some conducting substance. When such a communication is made, because the metallic coating touches the whole surface of the electrified glass, the whole quantity of redundant fluid easily passes from the side which was positively electrified to the other.

EXP. 1. Let a plate of glass, coated with tinfoil (except about  $1\frac{1}{2}$  inch from the edge) be charged, as described Prop. XVI. Exp. 5. Upon making a communication from one side to the other by a metallic rod, terminated with balls, there will be a sudden discharge.

2. Let



2. Let the same be done with a coated phial, or jar.—Such a phial is called the *Leyden Phial*.

3. Charge a jar coated on the inside with water, shot, or brass dust, and held on the outside by the hand, then discharge it in a dark room.

4. If two equal circular brass plates, one of which is suspended by a long metallic rod from the conductor parallel to the horizon, and the other, supported by a conductor, is placed parallel and opposite to the first, be electrified; the plate of air between them will be charged by the brass plates.

5. Let one coated jar be suspended by a wire under another; let the upper jar be charged by taking sparks from the conductor; the lower uninsulated jar will be charged with the fluid which passes from the side negatively electrified of the upper jar.

6. Discharge, in a dark room, a jar imperfectly coated.

#### P R O P. XIX.

A coated jar cannot be charged unless its outer surface be connected with some conductor.

For without such a conductor the fluid cannot pass from or to the outer surface, which is necessary in order to charge the jar.

Exp. An insulated jar cannot be charged.

#### P R O P. XX.

When a coated glass vessel is charged, the charge of electric fluid is in the glass, and not in the coating.

Exp. Lay a plate of glass between two metallic plates, as described Prop. XVI. Exp. 5. Having charged the plate of glass, remove the upper plate of metal by a glass handle; with some non-conducting substance, as silk, remove the electrified glass plate, and place it between two other plates of metal unelectrified and insulated; the plate of glass, thus coated afresh, will still be charged.

#### P R O P. XXI.

If the conductor be electrified positively, that side of the jar with which it has a communication will be electrified positively, the other, negatively.



EXP. 1. Charge one jar on the inside positively, and another negatively, and observe, in a dark room, the different appearances of the fluid, upon the point of a wire brought near to the ball which is connected with the inner side of each jar: when the point is presented to the jar positively electrified on the inner side, it will exhibit the appearance of a star; when presented to the other, that of a pencil.

2. Observe the different appearances, in a dark room, when with the same charged jar the point is presented towards the side positively, and towards the side negatively, electrified.

3. Between two jars, charged one negatively and the other positively, suspend by a silken string a cork ball, from which short threads hang freely; the ball will pass with a rapid motion from one to the other, and, being first attracted towards the jar positively electrified, then towards the other, it will receive the fluid from the former, and communicate it to the latter, till both are discharged. If both be charged in the same manner, the cork will remain at rest.

4. If after a jar is charged, the uncoated part of the jar be moistened by the breath, or by steam, the jar placed upon a conductor will be gradually discharged, and the fluid will be seen, in a dark room, to flash strongly from one side to the other: if the jar be insulated, the flashes will be greatest on the side positively electrified.

5. Let a discharging rod be applied without its balls to a charged jar, in such manner as to discharge the jar gradually: the point which approaches towards the side positively electrified, will, in a dark room, exhibit a star; the other point, a pencil.

6. Within the receiver of an air-pump <sup>place</sup> ~~let~~ two well polished brass balls, the lower supported on a brass stem by the plate of the pump, the other fixed on a stem which is moveable in the neck of the receiver: let the balls be brought within the distance of four or five inches from one another; then let the upper ball be connected with the conductor, and electrified positively: a lucid atmosphere will, in a dark room, appear on the lower surface of the upper ball: whereas if the upper ball be negatively electrified, the lucid atmosphere will be seen on the lower ball.

SCHOL. The last experiment establishes the theory of a single electric fluid: for if there were two contrary fluids, there must in this experiment be an atmosphere about each ball, attracting each other.

## P R O P. XXII.

The passing of the electric fluid from one side of a charged jar to the other, is apparently instantaneous, through whatever length of a metallic, or other good conductor, it is conveyed.

EXP. 1. Let a long wire, passing round a room, suspended by silk cords, be a part of the circuit of communication from one side of a charged jar to the other, the discharge will



will be apparently at the same instant in which the communication from one side to the other is completed.

2. Let any number of persons make a part of the circuit of communication ; the fluid will pass instantaneously through the whole circuit.

P R O P. XXIII.

The sudden discharge of a charged jar gives a painful sensation to any animal placed in the circuit of communication, called the *electric shock*.

Exp. Let some one receive the electric shock.

P R O P. XXIV.

If the circuit be interrupted, the fluid will become visible, and where it passes, it will leave an impression upon any intermediate body.

Exp. 1. Let the fluid pass through a chain, or through any metallic bodies placed at small distances from each other ; the fluid, in a dark room, will be visible between the links of the chain, or between the metallic bodies.

2. If the circuit be interrupted by several folds of paper, a perforation will be made through it, and each of the leaves will be protruded by the stroke from the middle towards the outward leaves.

3. Let a card be placed under wires which form the circuit, where the circuit is interrupted for the space of an inch ; the card will be discoloured. If one of the wires be placed under the card, and the other above it, the direction of the fluid may be seen.

4. If spirits of wine, or gun-powder, be made part of the circuit, it will be fired.

5. Inflammable air may be fired by an electric gun.

P R O P. XXV.

The force of the electric shock may be increased, by increasing the surface of coated glass.

If several coated jars, in which wires, terminated at the top by brass balls, are inserted, be placed in a wooden box lined with lead or tin, with which a metallic rod is connected ; an *electrical battery* is constructed.

Exp.



- Exp. 1. A battery being charged, a fine metallic wire brought into the circuit will be melted.
2. If a plain piece of metal be placed upon one of the rods of the discharger, and upon the other a needle with the point opposite to the surface of the metal, upon discharging the battery, the surface of the piece of metal will be marked with coloured circles, occasioned by thin *laminæ* of the metal raised in the explosion.
3. If a piece of leaf gold be put between two pieces of glass, and the whole fast bound together, the metal will be melted, and a metallic stain will be seen in both the glasses.
4. If a shock be sent through a needle, it will give it magnetic polarity.
5. An animal may be killed by being placed in the circuit of a battery.

## P R O P. XXVI.

The atmosphere is electrified, sometimes positively, and sometimes negatively.

Exp. Let a kite be sent up into the air with cord (consisting of copper thread twisted with twine); let the lower end of the cord be insulated by a silk line: a metallic conductor suspended from the lower end of the cord will be positively or negatively electrified.

## P R O P. XXVII.

The electric fluid and lightning are the same substance.

Their properties and effects are the same. Flashes of lightning are generally seen to form irregular lines in the air; the electric spark when strong has the same appearance. Lightning strikes the highest and most pointed objects; takes in its course the best conductor; sets fire to bodies; sometimes dissolves metals; rends to pieces some bodies; destroys animal life; in all which it agrees (as has been shewn) with the phenomena of electric fluid. Lastly, the lightning being brought from the clouds to an electrical apparatus, by a kite or wire, will exhibit all the appearances of the electric fluid.

## P R O P. XXVIII.

Buildings may be secured from the effects of lightning, by fixing a pointed iron rod higher than any part of the building, and continuing it, without interruption, to the ground, or the nearest water.

The electric fluid will, by means of the pointed rod, be gradually conveyed from the cloud to the earth by a continued stream, and thus prevent the effects of a sudden and violent explosion.

Exp.



EXP. Let a board, shaped like the gable end of a house, be fixed perpendicularly upon an horizontal board: in the perpendicular board let a hole be made, about an inch square and  $\frac{1}{4}$  inch deep; in this hole let a piece of wood nearly of the same dimensions be so inserted as to fall easily out of its place, and let a wire be fastened diagonally to this square piece of wood; let another wire, terminated by a brass ball, be fastened to the perpendicular board, with its ball above the board, and its lower end in contact with the diagonal wire in the square piece of wood; let the communication be continued by a wire to the bottom of the perpendicular board. If the wires in this state be made part of a circuit of communication, on discharging the jar the square piece of wood will not be displaced; but if the communication be interrupted by changing the direction of the diagonal wire, the square piece of wood will, upon the discharge, be driven out of its place.

If instead of the upper brass ball, a pointed wire be placed above the perpendicular board, the discharge may be drawn off without an explosion.

### P R O P. XXIX.

The electric fluid passes easily through a *vacuum*.

The air being a non-conductor, in proportion as it is removed, the effort of the electric fluid on the surface of the body positively electrified to pass to the next conductor, meets with less resistance, and therefore is diffused over a greater space.

EXP. 1. Let a jar be charged in *vacuo*.

2. Let a *luminous conductor* be placed in the circuit, and observe the fluid passing through it.

3. Let a *vacuum* be made a part of the circuit in discharging a phial.

4. Make a *vacuum* in a double barometer, and let the fluid pass from one leg to the other by connecting one of the vessels of mercury with the conductor.

5. The electric fluid may be made to pass through a large tube three feet in length, and four or five inches in diameter, if, being well exhausted, one end of it be connected with a large conductor.—The preceding experiments are to be performed in a dark room.

SCHOL. From the resemblance between these electrical appearances, and the atmospheric phenomena of the *Aurora Borealis*, meteors, &c. it is inferred, that these phenomena are produced by the electric fluid.

T H E E N D.



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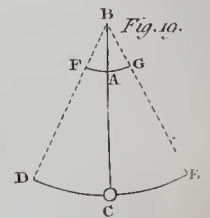
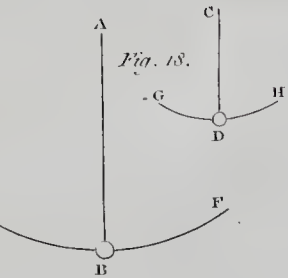
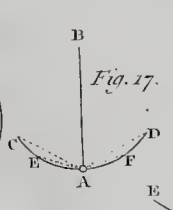
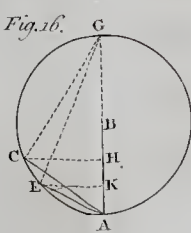
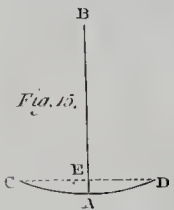
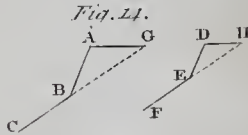
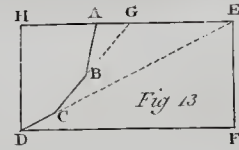
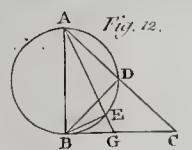
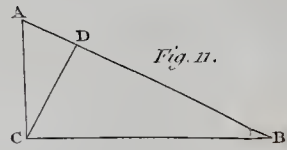
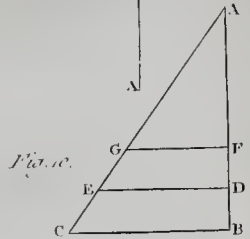
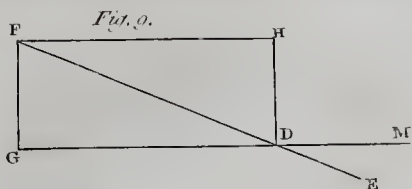
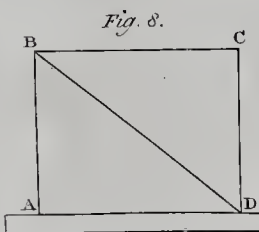
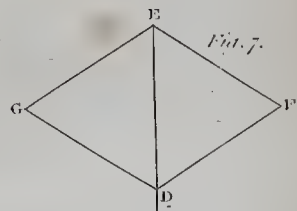
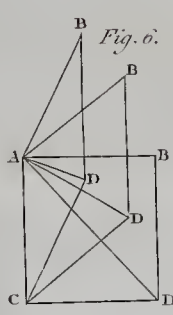
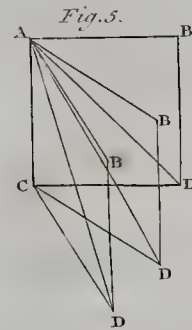
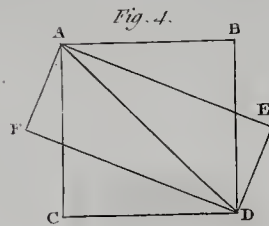
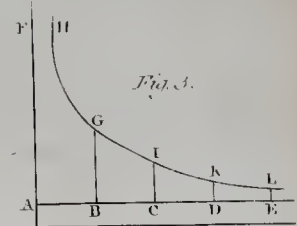
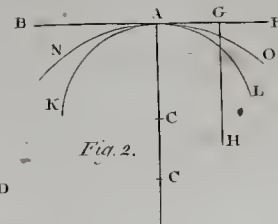
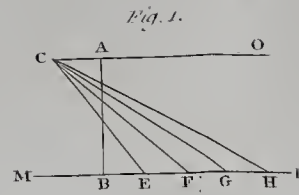
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Pl. I.

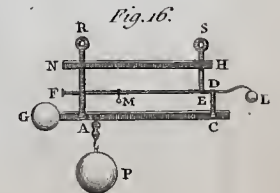
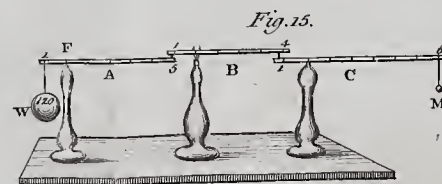
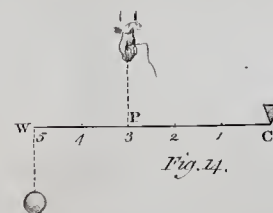
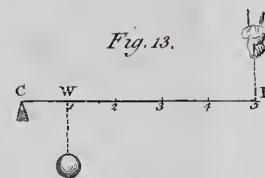
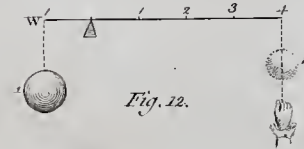
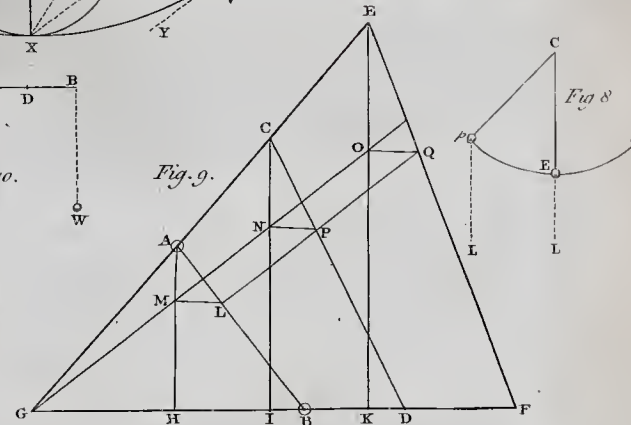
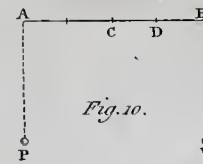
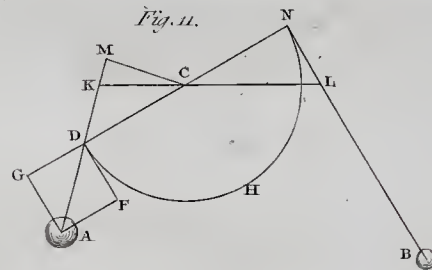
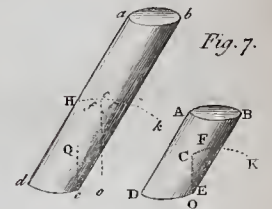
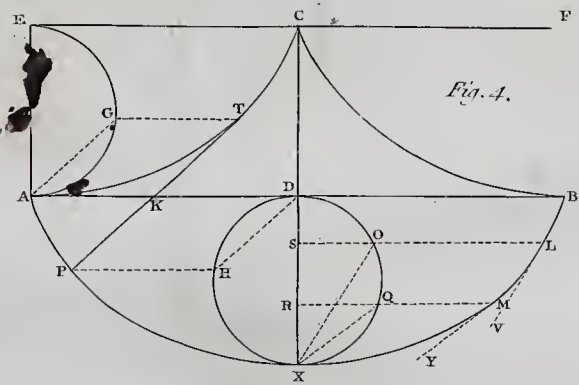
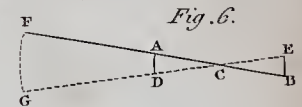
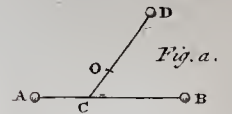
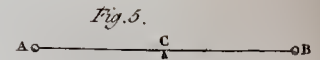
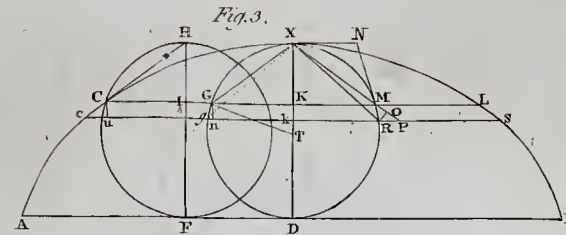
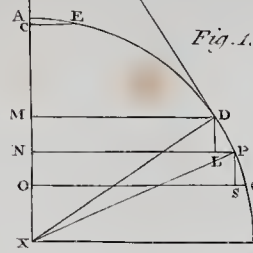
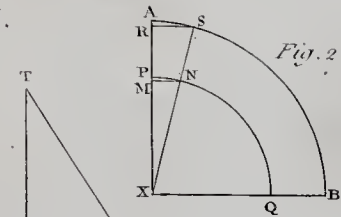








Pl. II.









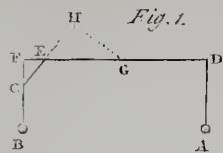


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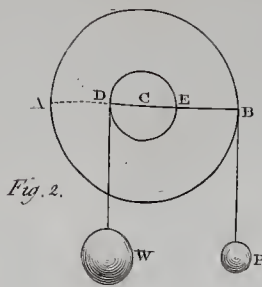
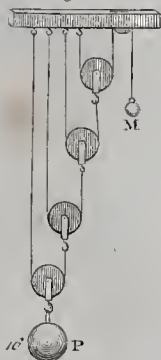


Fig. 2.



Fig. 3.

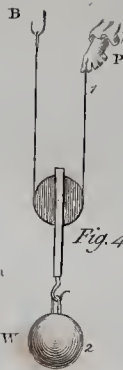


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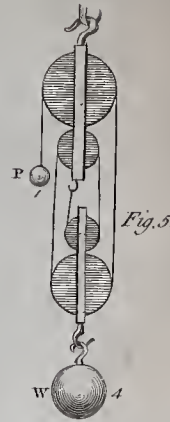


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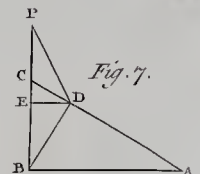


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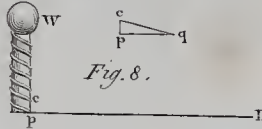


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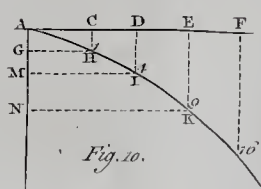


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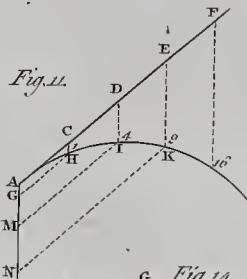


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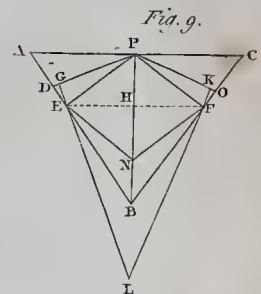


Fig. 9.

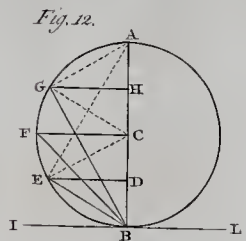


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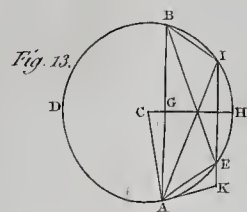


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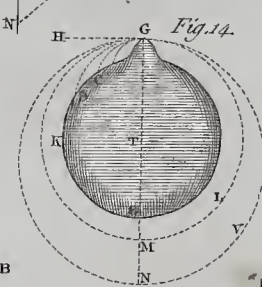


Fig. 14.

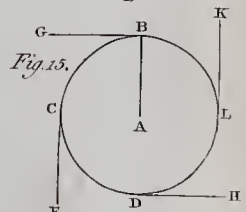


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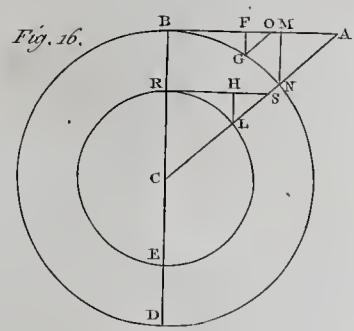


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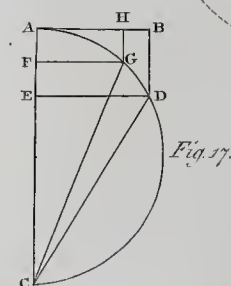


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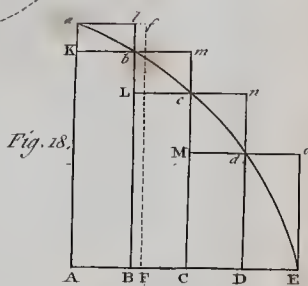


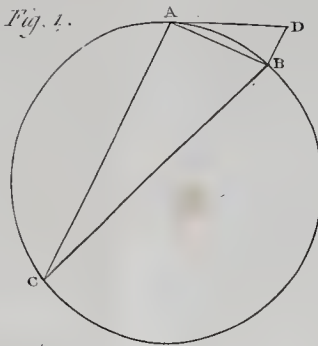
Fig. 18.



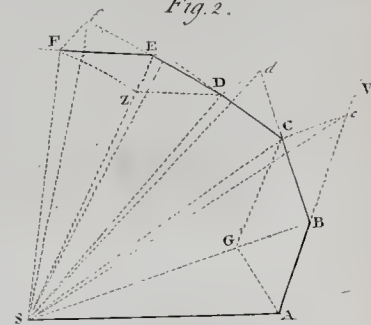




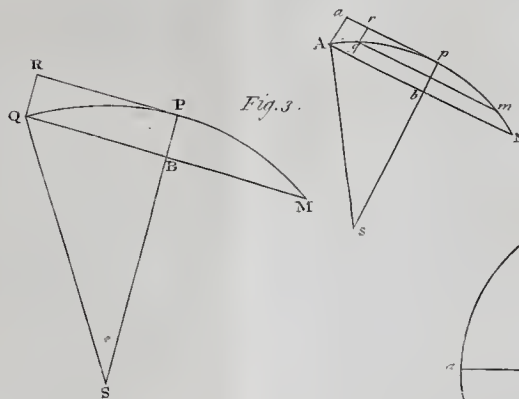
*Fig. 1.*



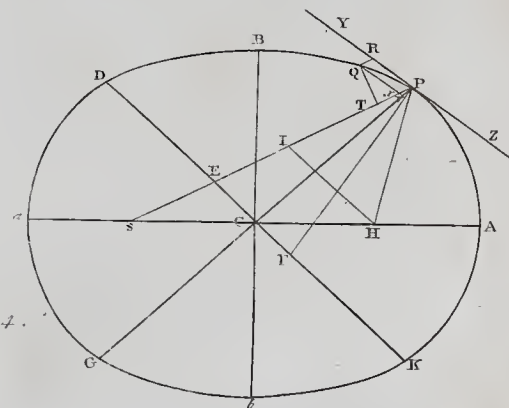
*Fig. 2.*



*Fig. 3.*



*Fig. 4.*

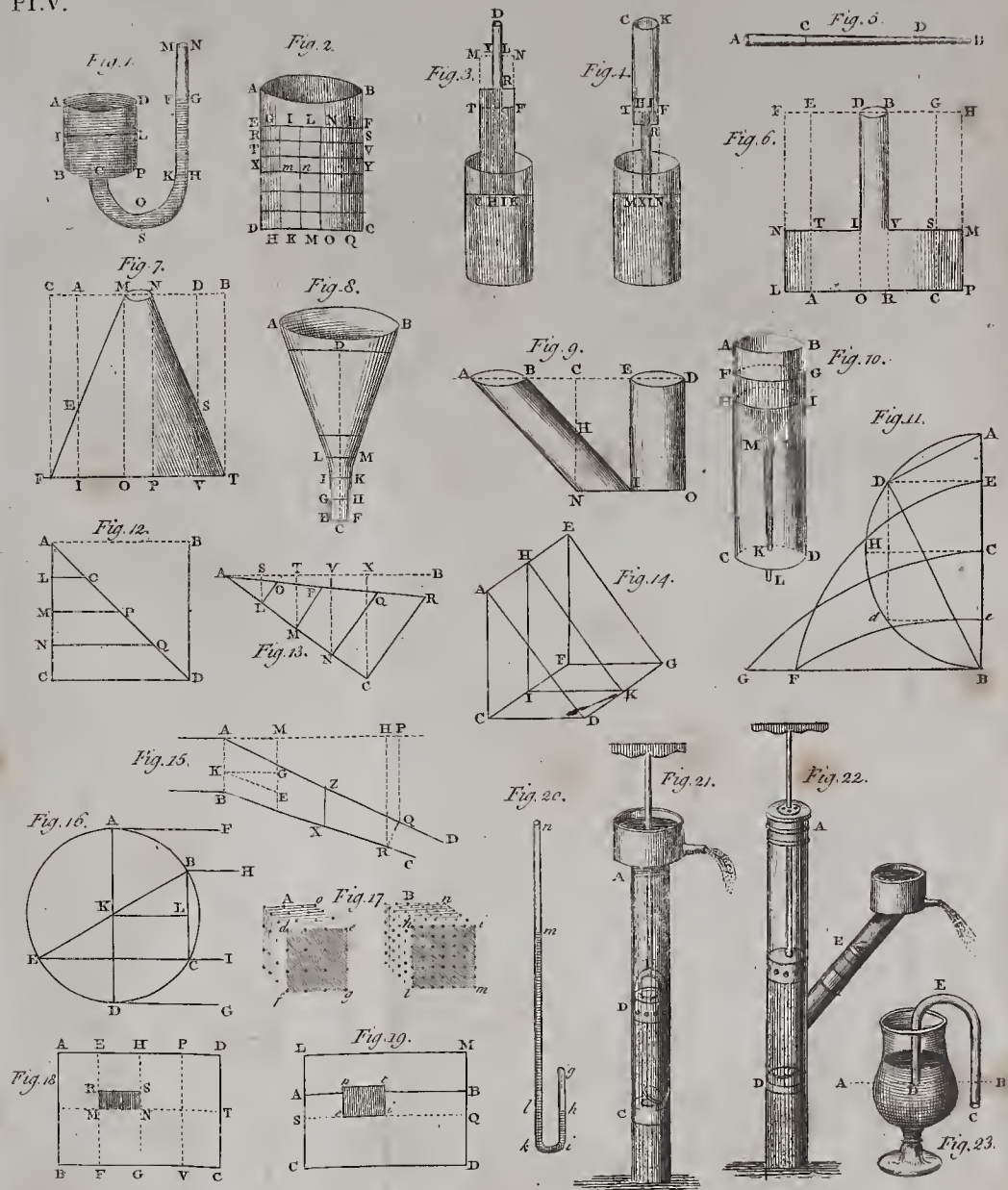








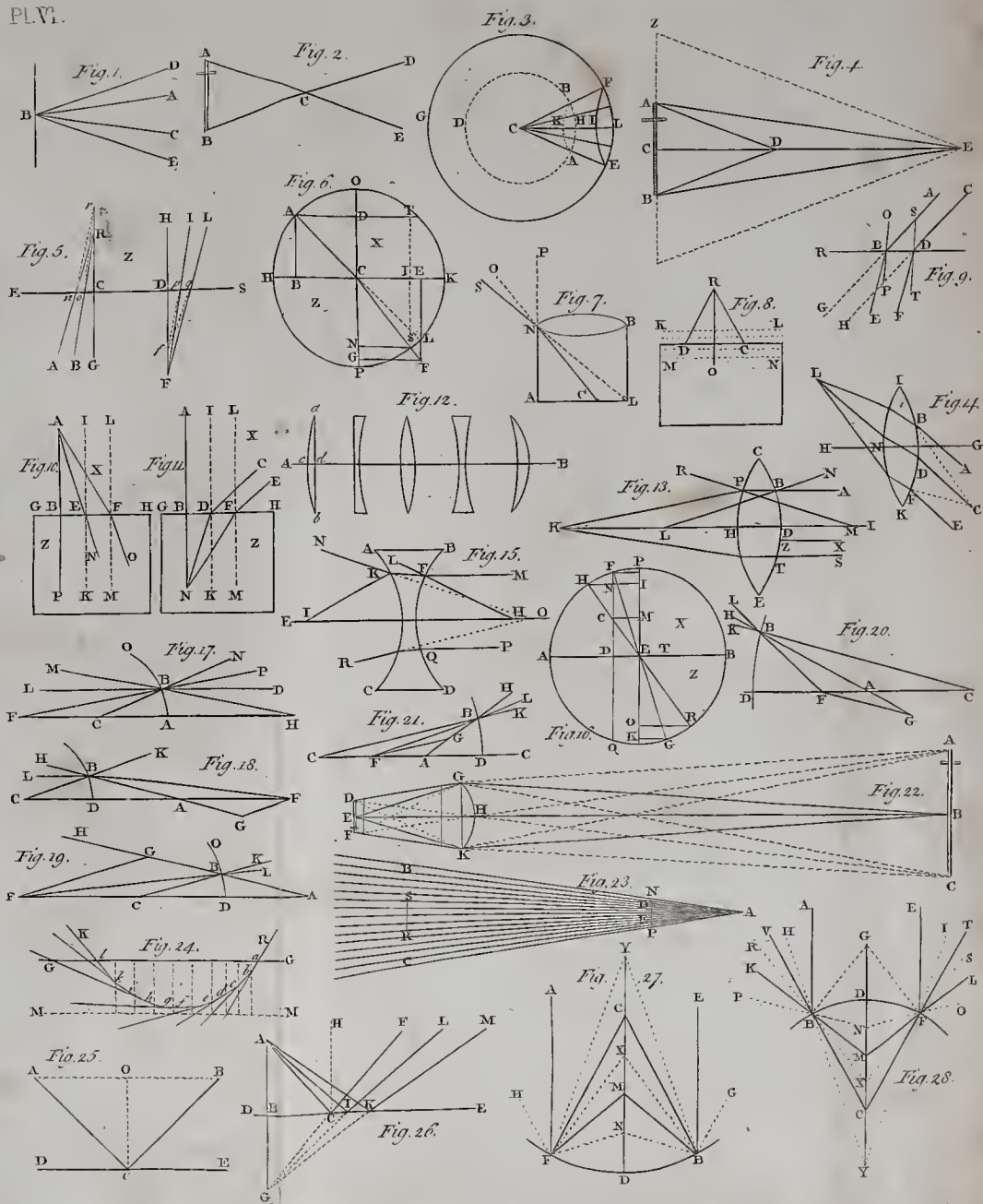
## Pl.V.









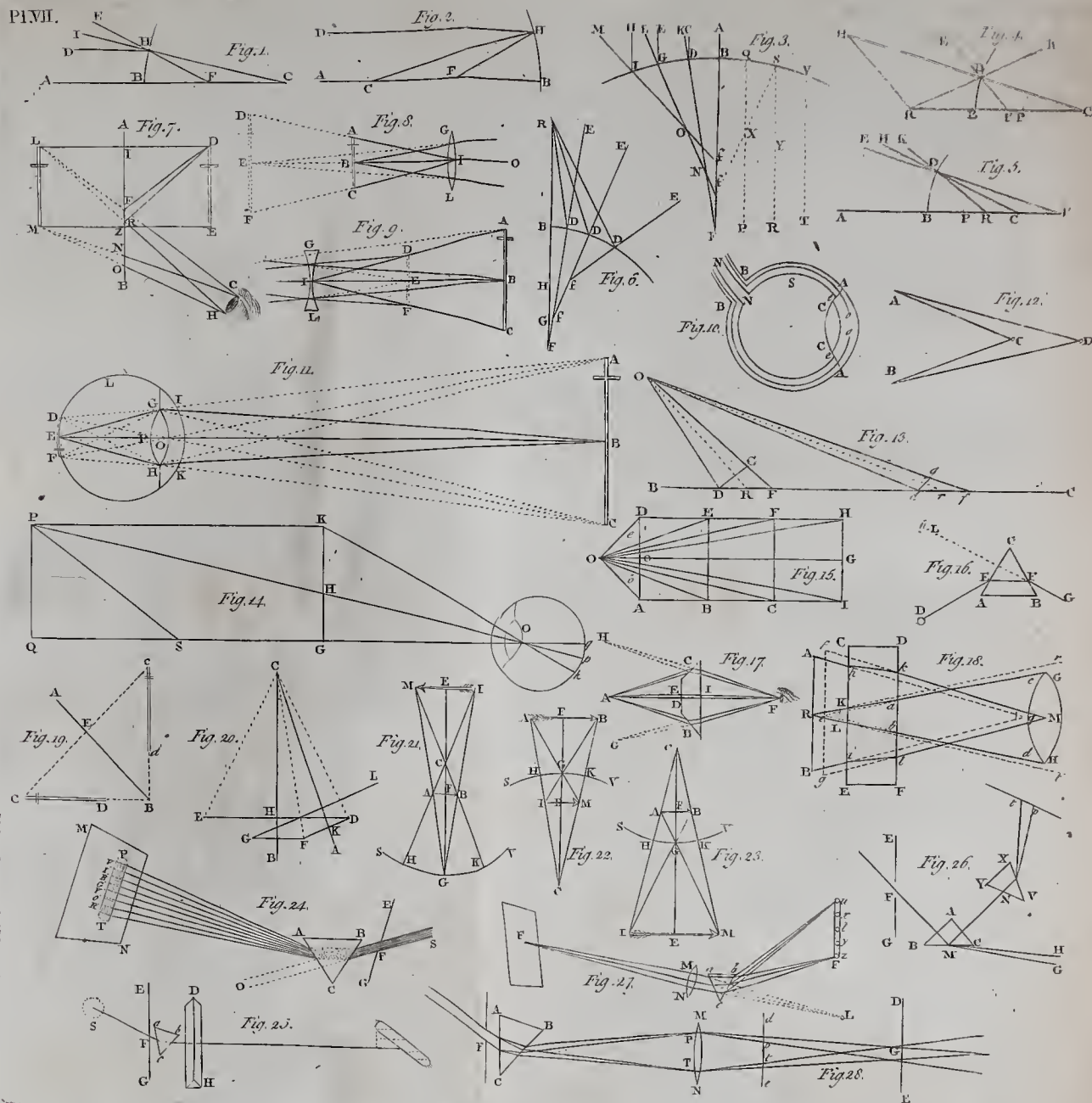








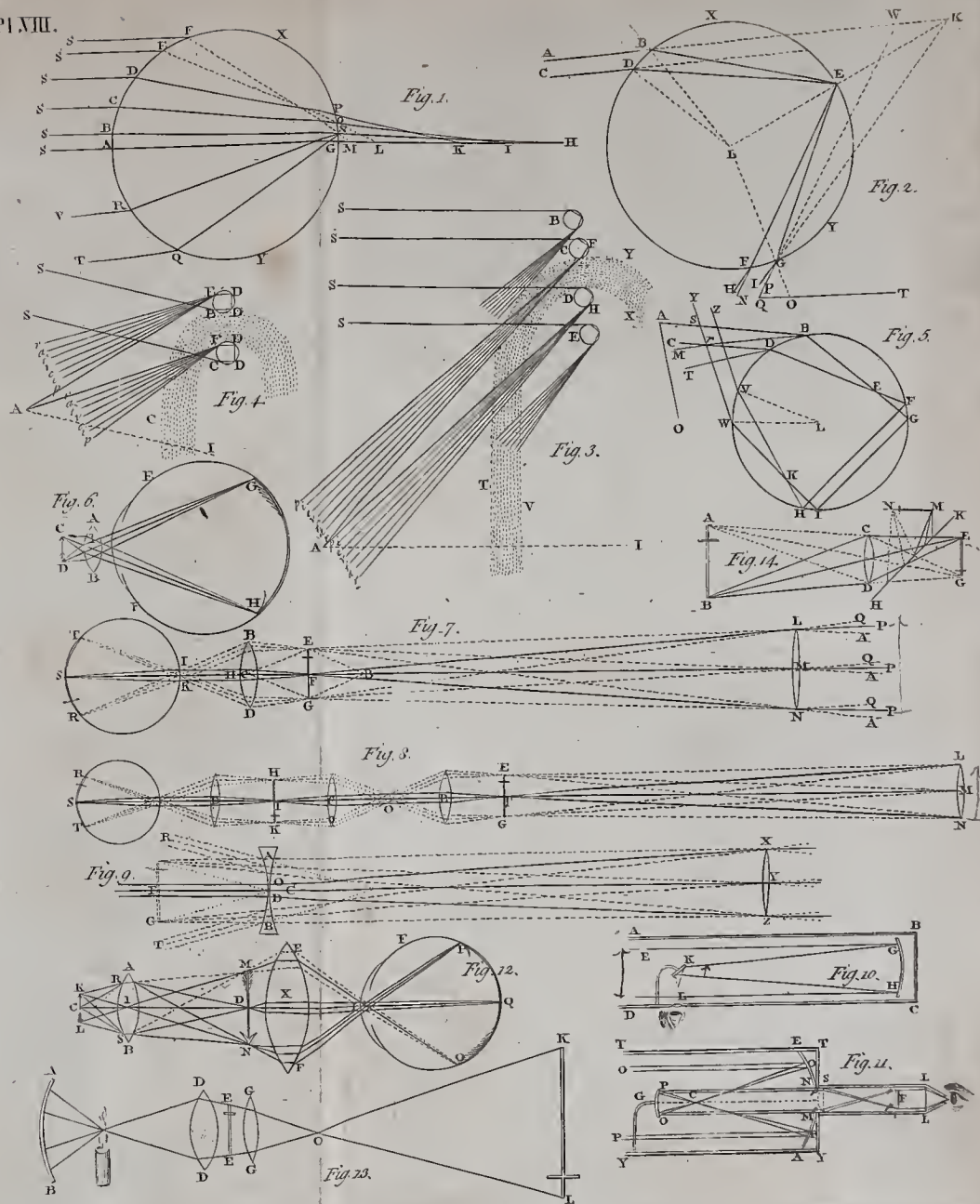
Pl.VII.







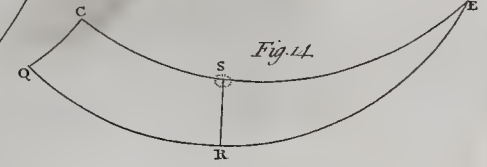
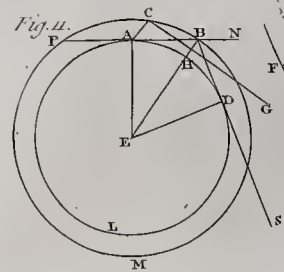
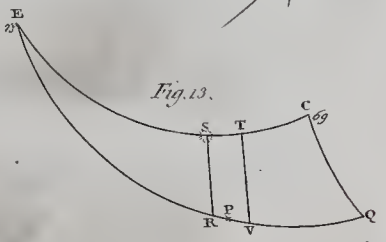
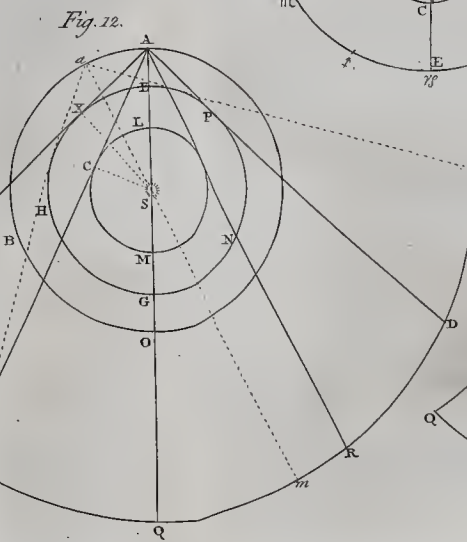
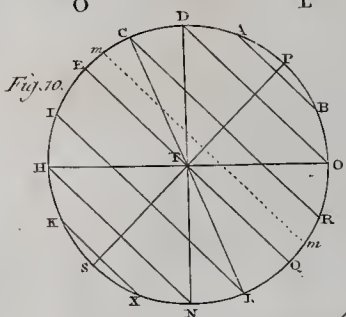
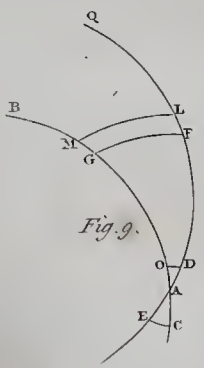
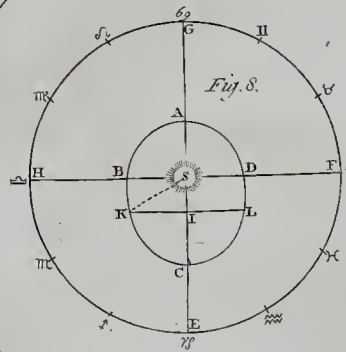
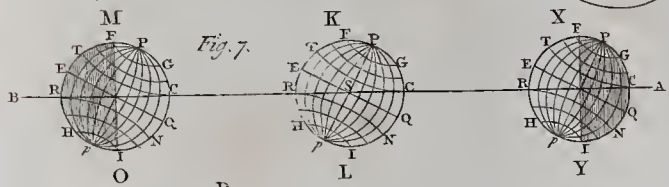
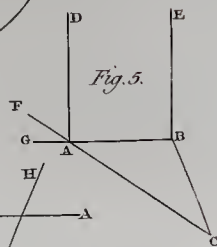
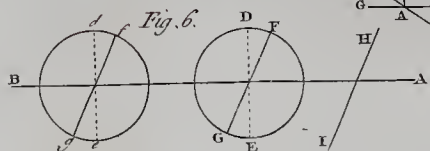
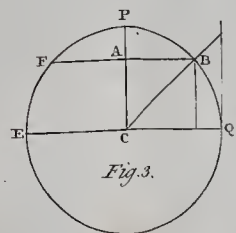
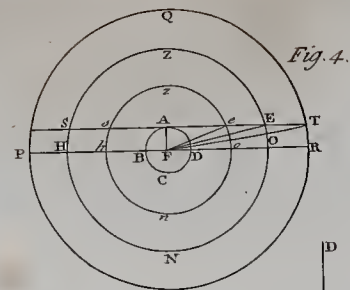
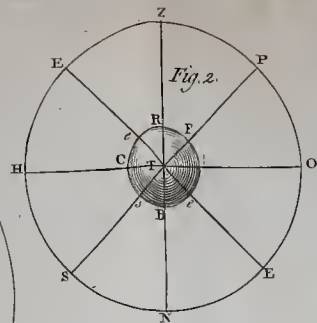
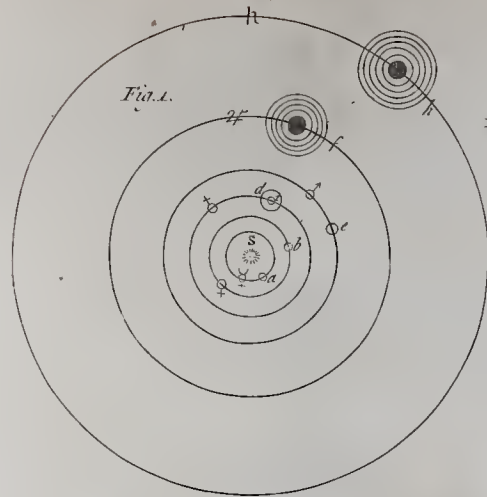










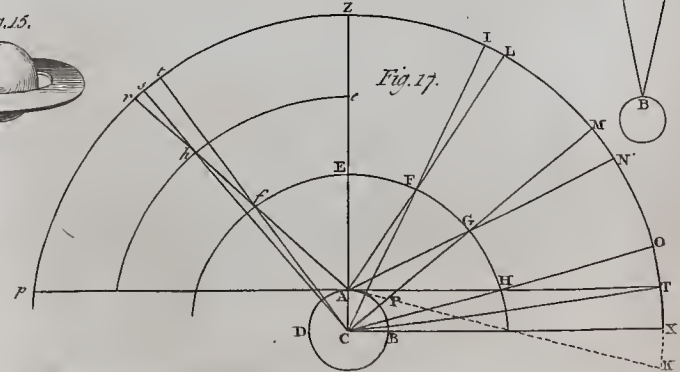
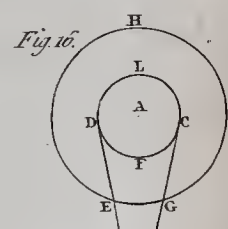
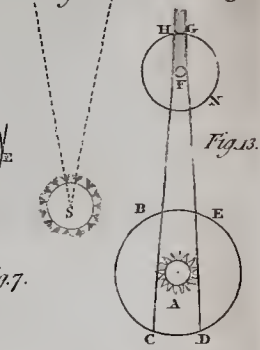
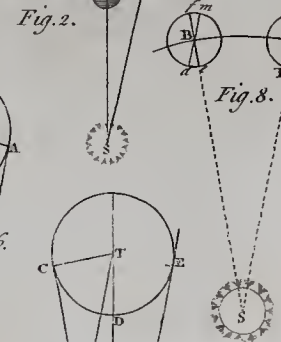
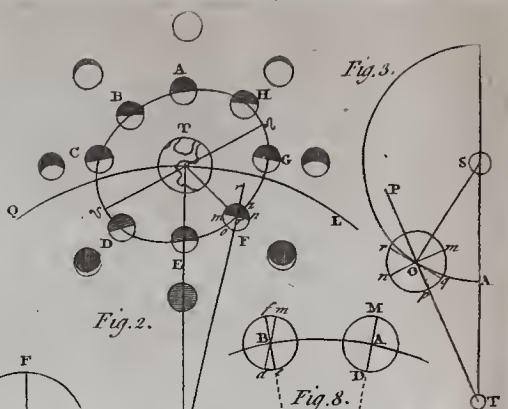
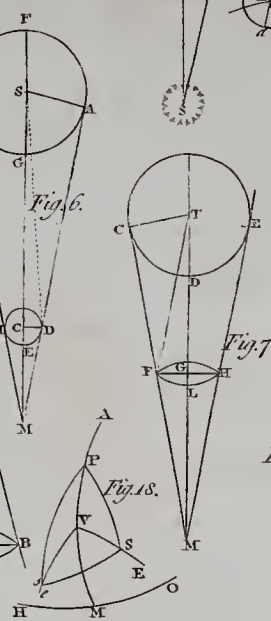
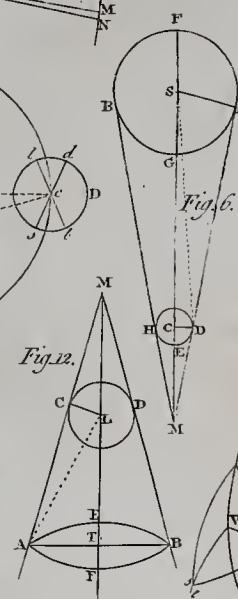
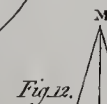
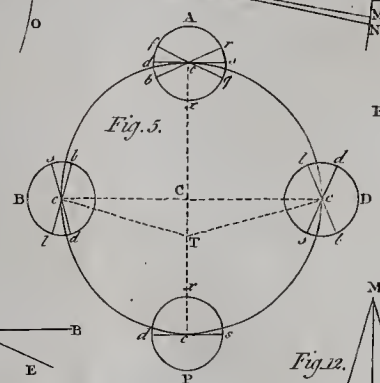
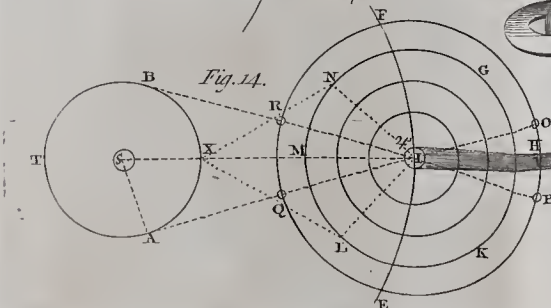
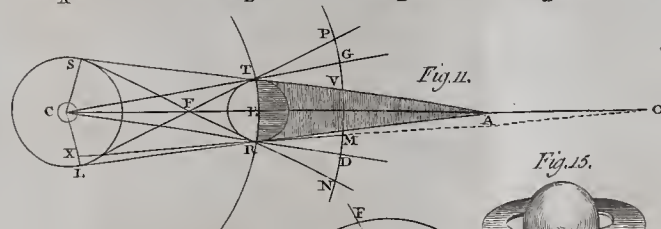
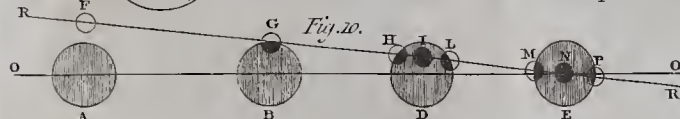
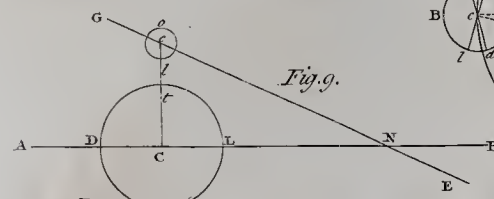
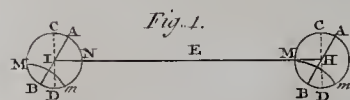
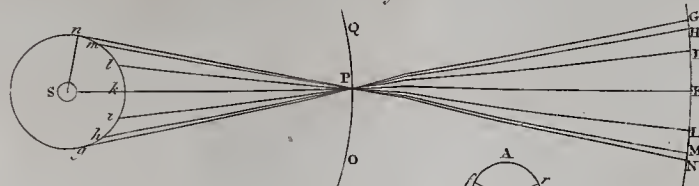
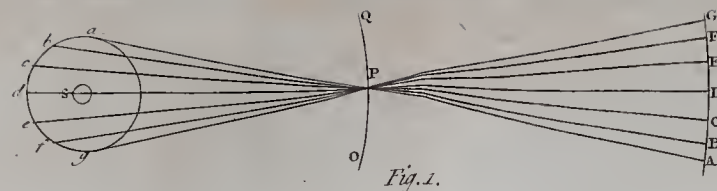








PLA.









PL. XL.

